QUANTUM ERGODICITY AND L-FUNCTIONS
LECTURE 4
SUBCONVEXITY

YIANNIS N. PETRIDIS

1. Preliminary facts about the Riemann zeta function

We recall that the Riemann zeta function

$$\zeta(s) = \frac{1}{n^s}$$

converges absolutely for $\Re(s) > 1$. Below we prove its analytic continuation in $\Re(s) > 0$ with pole at $s = 1$ and write it in a suitable way to prove Weyl's theorem.

Let $[x]$ denote the integral part of x and $\{x\}$ its fractional part. Using summation by parts we see that

$$\sum_{n \leq X} \frac{1}{n^s} = s \int_1^X \frac{[x]}{x^{s+1}} ds + \frac{[X]}{X^s}.$$

Since $\{x\} = x - [x]$ we write

$$\sum_{n \leq X} \frac{1}{n^s} = \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s \int_1^X \frac{\{x\}}{x^{s+1}} ds + \frac{1}{X^{s-1}} - \frac{\{X\}}{X^s}.$$

Letting $X \to \infty$ we deduce that for $\Re(s) > 1$ we have

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

Since the last integral converges absolutely for $\Re(s+1) > 1$, i.e. $\Re(s) > 0$, it defines a holomorphic function in this range. Moreover, we subtract the two equations above to deduce that

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + \frac{1}{(s-1)X^{s-1}} + \frac{\{X\}}{X^s} - s \int_X^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We apply it with $X = t^2$ and $\sigma = \Re(s) = 1/2$ to deduce that

$$\zeta(1/2 + it) = \sum_{n \leq t^2} \frac{1}{n^{1/2+it}} + O(1). \tag{1.1}$$

Remark. Eq. (1.1) is a way of approximating the Riemann zeta inside the critical strip by using a finite partial sum of the non-convergent series. There are many and better formulae of this type. The one worth mentioning is the approximate functional equation [3, 4.13, p. 79]. Let $\chi(s) = \pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$, i.e. the quotient of the Gamma factors in the functional equation of ζ. Then

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \chi(s) \sum_{n\leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma} \log |t|) + O(|t|^{1/2-\sigma} y^{\sigma-1})$$

for $0 < \sigma < 1$, $t > 0$, $2\pi xy = t$. This shows that to approximate $\zeta(1/2 + it)$ we need to finite sums of length approximately $\sqrt{t}$.

Remark. For all L-functions one can write an approximate functional equation (in a smooth form), see [2, p. 98]. This is of prime importance in trying to prove sub-convexity for it. We provide it here, although we will not use it. Let $G(u)$ be an even holomorphic function with $G(0) = 1$ and is bounded on the vertical strip $-4 < \Re(u) < 4$. Then for s in the vertical strip $0 \leq \Re(s) \leq 1$, we have

$$L(f, s) = \sum_n \frac{\lambda(n)}{n^s} V_s\left(\frac{n}{X\sqrt{q}}\right) + \epsilon(f, s) \sum_n \frac{\bar{\lambda}(n)}{n^{1-s}} V_{1-s}\left(\frac{nX}{\sqrt{q}}\right),$$

where q is the conductor, $\gamma(f, s)$ is the archimedean factor of the L-function, $\epsilon(f)$ is the root number,

$$V_s(y) = \frac{1}{2\pi i}\int_{(3)} y^{-u} G(u) \frac{\gamma(f, s+u)}{\gamma(f, s)} \frac{du}{u}, \quad \epsilon(f, s) = \epsilon(f) q^{1/2-s} \frac{\gamma(f, 1-s)}{\gamma(f, s)}.$$

2. Weyl's bound

In this section we study exponential sums to prove the celebrated result of Weyl:

Theorem 2.1 (Weyl and Hardy–Littlewood). *For the Riemann zeta function the following estimate holds on the critical line*

$$\zeta(1/2 + it) \ll t^{1/6} \log t, |t| \geq 2.$$

In the following $e(x) = e^{2\pi i x}$, $I = (a, b]$ and $||a|| = \min_{n\in\mathbb{Z}} |a - n|$.

Theorem 2.2 (Kuzmin–Landau, [1]). *If f is continuously differentiable, f' is monotonic and $||f'|| \geq \lambda > 0$ on I, then*

$$\sum_{n\in I} e(f(n)) \ll \lambda^{-1}.$$

Proof. We can assume that f' is increasing and that for some integer k we have

$$k + \lambda \leq f'(x) \leq k + 1 - \lambda.$$

Since $e(kn)=1$, we can assume that $\lambda \leq f'(x) \leq 1-\lambda$. With $g(n)=f(n+1)-f(n)$ we see that g is increasing (mean-value theorem) and that $\lambda \leq g(n) \leq 1-\lambda$. A simple calculation shows that

$$e(f(n)) = \frac{e(f(n)) - e(f(n+1))}{1-e(g(n))} = ((e(f(n)) - e(f(n+1)))c_n,$$

with

$$c_n = \frac{1}{2}(1 + i\cot \pi g(n)).$$

We use summation by parts (discrete version) to get

$$\sum_{n\in I} e(f(n)) = \sum_{n=a+1}^{b-1} e(f(n))(c_n - c_{n-1}) + e(f(a+1))c_{a+1} + e(f(b))(1-c_{b-1}),$$

$$\left|\sum_{n\in I} e(f(n))\right| \leq \frac{1}{2}\sum_{a+2}^{b-1} |\cot \pi g(n-1) - \cot \pi g(n)| + |c_{a+1}| + |1-c_{b-1}|.$$

Since cot is decreasing (notice that $g(n)\in[0,1]$), we see that the sum telescopes:

$$\left|\sum_{n\in I} e(f(n))\right| \leq \frac{1}{2}(\cot \pi g(a+1) - \cot \pi g(b-1)) + |\cot \pi g(a+1)| + |\cot \pi g(b-1)| + O(1).$$

We know that $\sin \pi x \geq 2x$ for $0 \leq x \leq 1/2$ (convexity of sin). Since $\sin \pi x = \sin \pi(1-x)$ we have $|\sin \pi x| \geq 2||x||$ and $\cot \pi x| \ll ||x||^{-1}$. This suffices. □

Theorem 2.3 (van der Corput)**.** *Suppose that f is real valued with two continuous derivatives on $I=(a,b]$. Moreover, suppose that for some constant $\lambda>0$ and some $\alpha \geq 1$ we have*

$$\lambda \leq |f''(x)| \leq \alpha\lambda, \quad x\in I.$$

Then

$$\sum_{n\in I} e(f(n)) \ll \alpha|I|\lambda^{1/2} + \lambda^{-1/2}.$$

Proof. See problem session and [1, p. 8]. □

Lemma 2.1 (Weyl differencing, [3])**.** *Let f be real valued, $a<n\leq b$ and $q\in\mathbb{N}$ with $q \leq b-a$. Then*

$$\left|\sum_{n\in I} e(f(n))\right| \ll \frac{b-a}{q^{1/2}} + \left(\frac{b-a}{q}\sum_{r=1}^{q-1}\left|\sum_{a<n\leq b-r} e(f(n+r)-f(n))\right|\right)^{1/2}.$$

Proof. We extend the function $e(f(x))$ to be 0 for $n \leq a$ and $n>b$. We rewriting the sum q times we see that

$$S = \sum_n e(f(n)) = \frac{1}{q}\sum_n \sum_{m=1}^{q} e(f(m+n)).$$

We note that the inner sum is 0, if $n \le a-q$ or $n > b-1$. We use the Cauchy–Schwartz inequality to get

$$\left|\sum_n e(f(n))\right| \le \frac{1}{q}\sum_n \left|\sum_{m=1}^{q} e(f(m+n))\right| \le \frac{1}{q}\left(\sum_n 1 \sum_n \left|\sum_{m=1}^{q} e(f(m+n))\right|^2\right)^{1/2}.$$

The values of n are at most $b-a+q \le 2(b-a)$ by our choice of q. So we can bound $\sum_n 1 \le 2(b-a)$. We deduce that

$$S \le \frac{1}{q}\left(2(b-a)\sum_n \left|\sum_{m=1}^{q} e(f(m+n))\right|^2\right)^{1/2}.$$

Here comes the main idea: the sum over m can be expanded and then we separate the diagonal and off-diagonal terms:

$$\left|\sum_{m=1}^{q} e(f(m+n)\right|^2 = \sum_{m,k=1}^{q} e(f(m+n)-f(k+n)) = q+2\sum_{1\le k<m\le q} e(f(m+n)-f(k+n)).$$

This gives

$$\sum_n \left|\sum_{m=1}^{q} e(f(m+n)\right|^2 \le 2(b-a)q + 2\left|\sum_n \sum_{k<m} e(f(m+n)-f(k+n))\right|. \tag{2.1}$$

The differences $f(m+n)-f(k+n)$ are quite often the same for different triples (m,k,n). With $1 \le k < m \le q$ we have that $m-k$ is between 1 and $q-1$. We set $r = m-k$ and set $d = k+n$ so that $f(m+n)-f(k+n) = f(d+r)-f(d)$. For given d and r we count how often this happens. The answer is that it happens $q-r$ times: $k=1, m=r+1$, $k=2, m=r+2$, $\ldots, k=q-r, m=q$. The absolute value in (2.1) is

$$\left|\sum_{r=1}^{q-1}(q-r)\sum_d e(f(d+r)-f(d))\right| \le q\sum_{r=1}^{q-1}\left|\sum_d e(f(d+r)-f(d))\right|.$$

We finally deduce that

$$S \le \frac{1}{q}\left(4(b-a)^2 q + 4(b-a)q\sum_{r=1}^{q-1}\left|\sum_d e(f(d+r)-f(d))\right|\right)^{1/2}.$$

□

Theorem 2.4. *Suppose that f is real valued with three continuous derivatives on $I = (a,b]$. Moreover, suppose that for some constant $\lambda > 0$ and some $\alpha \ge 1$ we have*

$$\lambda \le |f'''(x)| \le \alpha\lambda, \quad x \in I.$$

Then

$$\sum_{n\in I} e(f(n)) \ll |I|\alpha^{1/2}\lambda^{1/6} + |I|^{1/2}\lambda^{-1/6}.$$

Proof. The result is trivial if $\lambda \geq 1$.

We try to use van der Corput's result but we are given information on the third derivative. So we define for $r > 0$

$$g(x) = f(x+r) - f(x) \Longrightarrow g''(x) = f''(x+r) - f''(x) = rf'''(\xi),$$

by the mean value theorem, for some $\xi \in (x, x+r)$. With the given bound on the third derivative we have

$$r\lambda \leq |g''(x)| \leq r\alpha\lambda, \quad x \in I.$$

It follows from van der Corput's result above that

$$\sum_{a<n\leq b-r} e(g(n)) \ll |I|\alpha(r\lambda)^{1/2} + (r\lambda)^{-1/2}.$$

We now use the previous lemma on Weyl differencing to deduce that

$$\begin{aligned}\sum_{a<n\leq b} e(f(n)) &\ll |I|q^{-1/2} + \left(|I|q^{-1}\sum_{r=1}^{q-1}\left(|I|\alpha(r\lambda)^{1/2} + (r\lambda)^{-1/2}\right)\right)^{1/2} \\ &= |I|q^{-1/2} + (\alpha|I|^2q^{1/2}\lambda^{1/2} + |I|q^{-1/2}\lambda^{-1/2})^{1/2} \\ &= |I|q^{-1/2} + \alpha^{1/2}|I|q^{1/4}\lambda^{1/4} + |I|^{1/2}q^{-1/4}\lambda^{-1/4}.\end{aligned}$$

The first term is dominated by the second of $q = [\lambda^{-1/3}]$, as long as $\lambda \leq 1$ and $q \leq |I| = b - a$. This provides the bound

$$\sum_{a<n\leq b} e(f(n)) \ll \alpha^{1/2}|I|\lambda^{1/6} + |I|^{1/2}\lambda^{-1/6}.$$

The last case to consider is $q > |I|$. This gives $\lambda^{-1/3} \geq |I|$. Then $|I| \ll |I|^{1/2}\lambda^{-1/6}$ and the result follows from the trivial bound on the exponential sum. □

Proof of 2.1. We recall

$$\zeta(1/2+it) = \sum_{n\leq t^2} \frac{1}{n^{1/2+it}} + O(1). \tag{2.2}$$

We split the sum in three ranges (A): $n \leq t^{2/3}$, (B): $t^{2/3} < n \leq t$, and (C): $n > t$.

For (A) we use the function

$$f(x) = -\frac{1}{2\pi}t\log x \Longrightarrow f'''(x) = -\frac{t}{2\pi x^3}.$$

The previous theorem gives for $b \leq 2a$:

$$\sum_{a<n\leq b} n^{-it} \ll a(t/a^3)^{1/6} + a^{1/2}(t/a^3)^{-1/6} = O(a^{1/2}t^{1/6} + at^{-1/6}).$$

We can omit the second term if $a \le t^{2/3}$. By partial summation

$$\sum_{a<n\le b} \frac{1}{n^{1/2+it}} \ll t^{1/6}. \tag{2.3}$$

Using a dyadic decomposition we deduce that

$$\sum_{n\le t^{2/3}} \frac{1}{n^{1/2+it}} \ll t^{1/6}\log t.$$

For (B) we use van der Corput's theorem above to deduce that

$$\sum_{a<n\le b} n^{-it} \ll t^{1/2} + at^{-1/2}.$$

Summation by parts produces

$$\sum_{a<n\le b} \frac{1}{n^{1/2+it}} \ll (t/a)^{1/2} + (a/t)^{1/2}.$$

Therefore, even when $a > t^{2/3}$ but $a < t$, the estimate (2.3) still holds. Using a dyadic decomposition we deduce that

$$\sum_{t^{2/3}<n\le t} \frac{1}{n^{1/2+it}} \ll t^{1/6}\log t.$$

For (C) we can use Kuzmin–Landau to deduce that

$$\sum_{t<n\le t^2} \frac{1}{n^{it}} = O(1/t) \Longrightarrow \sum_{t<n\le t^2} \frac{1}{n^{1/2+it}} = O(1).$$

□

References

[1] Graham, S. W.; Kolesnik, G. van der Corput's method of exponential sums. London Mathematical Society Lecture Note Series, 126. Cambridge University Press, Cambridge, 1991. vi+120 pp. ISBN: 0-521-33927-8

[2] Iwaniec, H.; Kowalski, E. Analytic number theory. American Mathematical Society Colloquium Publications, 53. AMS, Providence, RI, 2004. xii+615 pp. ISBN: 0-8218-3633-1

[3] Titchmarsh, E. C. The theory of the Riemann zeta-function. Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986. x+412 pp. ISBN: 0-19-853369-1

Department of Mathematics, University College London, Gower Street, London WC1E 6BT

Email address: i.petridis@ucl.ac.uk