QUANTUM ERGODICITY AND L-FUNCTIONS
LECTURE 3
QUE FOR EISENSTEIN SERIES

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1. What is Quantum Ergodicity? Some generalities

Negatively curved manifolds M have ergodic geodesic flow [1]. We will not need a precise definition here. We are interested in the reflection of this property on the quantum system underlying the manifold. All hyperbolic manifolds are of interest but we restrict our attention to the simplest one: the modular surface $M = \Gamma\backslash\mathbb{H}$, where $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Quantum chaology studies how the quantum behaviour of the system reflects the dynamic properties of the system, in our case the geodesic flow. The corresponding quantum object is a differential operator, in this case $\hbar^2\Delta$, where Δ is the Laplace–Beltrami operator. In quantum mechanics the movement of a free particle is prescribed by the solution of the Schrödinger equation (time dependent)

$$i\hbar\frac{\partial\psi}{\partial t} = (\hbar^2\Delta + V)\psi.$$

By separation of variables, we end up having to study the eigenvalue equation

$$\Delta u_n + \lambda_n u_n = 0,$$

where the energy of the system is $E_n = \lambda_n\hbar^2/2$. We express in simple mathematical terms the main problem of quantum chaos: study the behaviour of the following measures $d\mu_n$ defined by:

$$\mu_n(A) = \int_A |u_n|^2\, d\mu,$$

as $\lambda_n \to \infty$ and for nice sets A in the manifold. Here $d\mu$ is the volume measure on the manifold. We need somehow to normalise these measures: we assume that the eigenfunctions u_j have L^2-norm equal to one. We are interested in the equidistribution of these measures. This property is called quantum ergodicity. Schnirelman [9], Colin de Verdière [2], and Zelditch [12] have proved this equidistribution for 'almost' all eigenfunctions: there exists a full density subsequence λ_{j_k}, i.e. $\sum_{\lambda_{j_k}\le\lambda} 1 \sim \sum_{\lambda_n\le\lambda} 1$, such that

$$d\mu_{j_k} \to \frac{1}{\mathrm{vol}(M)}d\mu, \quad k \to \infty.$$

Rudnick and Sarnak formulated a more precise conjecture: the density one sequence of eigenfunctions should in fact be *all* eigenfunctions, i.e. there are not exceptional subsequences, on which Quantum Ergodicity fails to hold, at least in negative curvature. This means

$$\text{(QUE)} \qquad d\mu_n \to \frac{1}{\mathrm{vol}(M)} d\mu.$$

This conjecture is called Quantum Unique Ergodicity (QUE). It has been proved in a limited number of cases: one of them is the modular surface by work of Lindenstrauss [6] and Soundararajan [10]. These are theorems of high complexity. We restrict here to the continuous spectrum of the modular surface and prove QUE for Eisenstein series. Here the analysis is simpler and highlights easily the relation with the subconvexity problem for L-functions.

Remark. There is a more difficult notion of quantum ergodicity that takes place on the tangent bundle of the space.

2. QUE for Eisenstein series

We denote $d\mu(z)$ the hyperbolic volume element, i.e. $d\mu(z) = y^{-2}dx\, dy$. We also define the measures

$$d\mu_t(z) = |E(z, 1/2 + it)|^2 \, d\mu(z).$$

We observe that, as $E(z, 1/2 + it)$ is not in $L^2(\Gamma\backslash\mathbb{H})$, we have $\mu_t(\Gamma\backslash\mathbb{H}) = \infty$. This means that we cannot normalize the measures $d\mu_t$ in a canonical way i.e. L^2-normalized. However, if we restrict to 'nice sets' then we can compute the asymptotic behaviour of $d\mu_t$ as $t \to \infty$.

Theorem 2.1 (QUE for Eisenstein series, [7])**.** *Let A and B be compact Jordan measurable subsets of $\Gamma\backslash\mathbb{H}$, then we have*

$$\lim_{t\to\infty} \frac{\mu_t(A)}{\mu_t(B)} = \frac{\mathit{area}(A)}{\mathit{area}(B)}.$$

In fact

$$\mu_t(A) \sim \frac{6}{\pi} \mathit{area}(A) \log t \sim \frac{\mathit{area}(A)}{\mathit{area}(\Gamma\backslash\mathbb{H})} \log(1/4 + t^2), \quad t \to \infty.$$

This theorem is almost immediate consequence of the continuous version below.

Theorem 2.2 ([7])**.** *For $\psi \in C_0(\Gamma\backslash\mathbb{H})$, i.e. ψ continuous of compact support on the modular surface we have*

$$\int_{\Gamma\backslash\mathbb{H}} \psi(z) d\mu_t(z) \sim \frac{1}{\mathit{area}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} \psi(z) d\mu(z) \cdot \log(1/4 + t^2), \quad t \to \infty.$$

Notice that we write the theorem slightly differently from [7]. One reason is to show the eigenvalue $1/4 + t^2$ of the Eisenstein series. We also explicitly show the hyperbolic volume of the modular surface, which is $\pi/3$. We remark that due to small errors in the calculation, their constant in the theorem is wrong: they find $24/\pi$, while the correct is $3/\pi$. We will follow the proof in [7]. We will explain how it is reduced to certain estimates for L-functions, in particular, for $\zeta(s)$ and $L(u_j, s)$. The idea is to look at the individual components in the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$, i.e. Maaß cusp forms and incomplete Eisenstein series. If we understand the behaviour on the measures $d\mu_t$ on these, we have essentially proved the result.

2.1. Contribution of Maaß cusp forms.

Proposition 2.1. *Let u_j be a Hecke–Maaß cusp form, i.e. cusp form that is also eigenfunction of the Hecke operators. Then*

$$\int_{\Gamma\backslash\mathbb{H}} u_j(z)d\mu_t(z) \ll_{j,\epsilon} |t|^{-1/6+\epsilon}.$$

This follows from the estimate

$$\int_{\Gamma\backslash\mathbb{H}} u_j d\mu_t(z) \ll_j \frac{\log^2 t}{\sqrt{t}} \left|L(u_j, 1/2+2it)\right|,$$

and Meurman's subconvexity result [8] *(see section below) on the L-function of a Maaß cusp form:*

$$L(u_j, 1/2+it) \ll_j t^{1/3}, \quad t\to\infty.$$

Proof. We set

$$J_j(t) = \int_{\Gamma\backslash\mathbb{H}} u_j d\mu_t(z) = \int_{\Gamma\backslash\mathbb{H}} u_j E(1/2+it)E(z,1/2-it)d\mu(z),$$

since $E(z,s) = \overline{E(z,\bar{s})}$ for $\Re(s) > 1$ and by analytic continuation for all s. This prompts us to consider

$$I_j(s) = \int_{\Gamma\backslash\mathbb{H}} u_j(z)E(z,1/2+it)E(z,s)d\mu(z)$$

and set $s = 1/2 - it$ at the end. All the integrals here are convergent, as u_j is rapidly decreasing at the cusp. To understand $I_j(s)$ we unfold à la Rankin–Selberg to get

$$I_j(s) = \int_0^\infty \int_0^1 u_j(z)E(z,1/2+it)y^s dx\, \frac{dy}{y^2}.$$

Remark. This calculation is valid for all Maaß cusp forms. If u_j is odd, since $E(z,1/2+it)$ is even, the integral is zero, and there is no need for extra calculation. So we can assume that u_j even.

We use the Fourier expansion of u_j (Lecture 2) and of $E(z,s)$ (Lecture 1, (2.5)). We integrate in x. We get for $\Re(s) > 1$

$$I_j(s) = \frac{2\rho_j(1)}{\xi(1+2it)} \sum_1^\infty \lambda_j(n)n^{it}\sigma_{-2it}(n) \int_0^\infty K_{it_j}(2\pi ny)K_{it}(2\pi ny)y^{s-2}dy.$$

We now change variable $2\pi ny \to y$ and use the Mellin transform of the product of two K-Bessel functions (see [4, p.205])

$$\int_0^\infty K_\mu(y)K_\nu(y)y^{s-1}dy = 2^{s-3}\Gamma(s)^{-1}\prod\Gamma\left(\frac{s\pm\mu\pm\nu}{2}\right)$$

for $\Re(s) > |\Re(\mu)| + |\Re(\nu)|$ and with all four choices of $\pm$ in the product. We get

$$I_j(s) = \frac{2\rho_j(1)}{2^s\pi^s\xi(1+2it)}\sum_1^\infty\frac{\lambda_j(n)n^{it}\sigma_{-2it}(n)}{n^s}\int_0^\infty K_{it_j}(y)K_{it}(y)y^{s-2}dy$$

$$= \frac{\rho_j(1)}{4\xi(1+2it)}\frac{\Gamma\left(\frac{s+it_j+it}{2}\right)\Gamma\left(\frac{s-it_j+it}{2}\right)\Gamma\left(\frac{s+it_j-it}{2}\right)\Gamma\left(\frac{s-it_j-it}{2}\right)}{\pi^s\Gamma(s)}R_j(s),$$

where

$$R_j(s) = \sum_1^\infty\frac{\lambda_j(n)n^{it}\sigma_{-2it}(n)}{n^s}.$$

We now use the Exercise 4 from Problem sheet 1 to deduce that

$$R_j(s) = \frac{1}{\zeta(2s)}L(u_j, s-it)L(u_j, s+it).$$

This implies that

$$\begin{aligned}J_j(t) =&\rho_j(1)\pi^{-2it}\frac{|\Gamma(1/4+it_j/2)|^2}{4|\zeta(1+2it)|^2}\frac{\Gamma(1/4-it_j/2-it)\Gamma(1/4+it_j/2-it)}{|\Gamma(1/2+it)|^2}\\ &\cdot L(u_j, 1/2)L(u_j, 1/2-2it).\end{aligned}$$

By Stirling's formula [3, 8.328, p 895] we have

$$|\Gamma(x+it)| \sim \sqrt{2\pi}e^{-\pi|t|/2}|t|^{x-1/2}, \quad |t| \to \infty. \tag{2.1}$$

This implies for $t > 0$ and for the Gamma factors the estimate

$$\frac{\Gamma(1/4-it_j/2-it)\Gamma(1/4+it_j/2-it)}{|\Gamma(1/2+it)|^2} \ll \frac{(e^{-\pi t/2}t^{1/4-1/2})^2}{(e^{-\pi t/2}t^{1/2-1/2})^2} = t^{-1/2}.$$

We note that if $L(u_j, 1/2) = 0$, there is nothing to estimate. We need to deal with $\zeta(1+2it)^{-1}$. The proof of the prime number theorem due to Hadamard and de la Vallée Poussin gives in fact, see [11, 3.11.8]

$$\frac{1}{\zeta(1+it)} \ll \log t.$$

This suffice to prove Proposition 2.1.

□

Remark. There is a weaker but easier to prove estimate for $\zeta(s)$ on the line of convergence:

$$\frac{1}{\zeta(1+it)} \ll \log^7 t,$$

which uses a smaller zero-free region for $\zeta(s)$. This is proved in [11, 3.6.5]. While not the best, it still suffices to prove

$$\int_{\Gamma\backslash\mathbb{H}} u_j d\mu_t \ll |t|^{-1/6+\epsilon}.$$

2.2. **Contribution of incomplete Eisenstein series.** We aim to prove the following proposition.

Proposition 2.2. *Let $\psi(y)$ be smooth and rapidly decaying at ∞ and 0. Then*

$$\int_{\Gamma\backslash\mathbb{H}} E(\psi,z)d\mu_t(z) \sim \frac{3}{\pi}\int_0^\infty \psi(y)\frac{dy}{y^2}\cdot\log(1/4+t^2), \quad t\to\infty.$$

Remark. Unfolding à la Rankin–Selberg gives

$$\int_{\Gamma\backslash\mathbb{H}} E(\psi,z)d\mu(z) = \int_0^\infty \psi(y)\frac{dy}{y^2}.$$

Warning: In the calculation below t is not the real part of s.

The incomplete Eisenstein series $E(\psi,z)$ is smooth and rapidly decreasing at the cusp. We evaluate it on the measure $d\mu_t$. Unfolding, using (2.5) from Lecture 1 and Parceval we get

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} E(\psi,z)d\mu_t(z) = &\int E(\psi,z)\left|E(z,1/2+it)\right|^2 d\mu(z)\\ =&\frac{1}{2\pi i}\int_{\Gamma\backslash\mathbb{H}}\int_{\Re(s)=2}\hat\psi(s)E(z,s)ds\left|E(z,1/2+it)\right|^2 d\mu(z)\\ =&\frac{1}{2\pi i}\int_0^\infty\int_{\Re(s)=2}\hat\psi(s)y^s ds\int_0^1\left|E(z,1/2+it)\right|^2 d\mu(z)\\ =&\frac{1}{2\pi i}\int_0^\infty\int_{\Re(s)=2}\hat\psi(s)y^s ds\left(\left|y^{1/2+it}+\phi(1/2+it)y^{1/2-it}\right|^2\right.\\ &\left.+\frac{8}{|\xi(1+2it)|^2}\sum_{n=1}^\infty y\left|\sigma_{-2it}(n)\right|^2\left|K_{it}(2\pi ny)\right|^2\right)\frac{dy}{y^2}.\end{aligned}$$

Notice that the Fourier series of $E(z,1/2+it)$ is even, which explains the factor 8. Recall that the scattering function $\phi(s)$ satisfies $\phi(s)\phi(1-s)=1$, using $\phi(s)=\xi(2-2s)/\xi(2s)$. This implies that $|\phi(1/2+it)|=1$. Using Mellin inversion, we see that the contribution of the zeroth Fourier coefficient in the integral above becomes

$$2\frac{1}{2\pi i}\int_0^\infty \hat\psi(s)y^{s+1}\frac{dy}{y^2}+g(t) = 2\int_0^\infty \psi(y)dy/y+g(t,)$$

where

$$g(t) = 2\frac{1}{2\pi i}\int_0^\infty \hat{\psi}(s)y^{s+1}\Re(\phi(1/2+it)y^{2it})\frac{dy}{y^2}.$$

By Mellin inversion and successive integrations by parts for the smooth function $\psi(y)$ we see that for all $A>0$

$$\int_0^\infty \psi(y)y^{2it}\frac{dy}{y} = O(t^{-A}).$$

We summarise the result:

Lemma 2.1.

$$\int_{\Gamma\backslash\mathbb{H}} E(\psi,z)d\mu_t(z) = 2\int_0^\infty \psi(y)\frac{dy}{y} + I_2(t) + O(t^{-A}),$$

where

$$I_2(t) = \frac{8}{2\pi i|\xi(1+2it)|^2}\int_{\Re(s)=2}\hat{\psi}(s)\sum_{n=1}^\infty \frac{|\sigma_{-2it}(n)|^2}{n^s}\int_0^\infty |K_{it}(2\pi y)|^2 y^s\frac{dy}{y}ds.$$

The series was evaluated by Ramanujan and the result from Problem Sheet 1 gives

$$Z(s,t) = \sum_{n=1}^\infty \frac{|\sigma_{-2it}(n)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s-2it)\zeta(s+2it)}{\zeta(2s)}.$$

As far as the integral of the K-Bessel functions we proceed as above (see [4, p.205]) to get

$$\tilde{\gamma}(s,t) = \int_0^\infty |K_{it}(2\pi y)|^2 y^s\frac{dy}{y} = \frac{\Gamma^2(s/2)\Gamma(s/2-it)\Gamma(s/2+it)}{8\pi^s\Gamma(s)}.$$

Therefore,

$$I_2(t) = \frac{8}{2\pi i|\xi(1+2it)|^2}\int_{\Re(s)=2}\hat{\psi}(s)Z(s,t)\tilde{\gamma}(s,t)ds.$$

2.3. **Evaluation of $I_2(t)$.** We shift the contour of integration from the line $\Re(s)=2$ to $\Re(s)=1/2$. We register the result in the lemma below.

Lemma 2.2. *We have*

$$I_2(t) = \frac{8}{|\xi(1+2it)|^2}\mathrm{Res}_{s=1}\hat{\psi}(s)Z(s,t)\tilde{\gamma}(s,t) + O\left(\frac{\log^2 t}{t^{1/2}}\cdot\max_{|u-2t|<t^\epsilon}|\zeta(1/2+iu)|^2\right) + O(t^{-A}).$$

Proof. We set

$$f(s,t) = \frac{8}{|\xi(1+2it)|^2}\hat{\psi}(s)Z(s,t)\tilde{\gamma}(s,t).$$

We easily see that $Z(s,t)$ has a double pole at $s=1$ and simple poles at $1\pm 2it$ for $\Re(s)\geq 1/2$. Using Stirling's formula we see that $\tilde{\gamma}(s,t)$ grows at most polynomially as a function of s on vertical lines. The same is true for $Z(s,t)$, since $\zeta(s)$ does the same (this is a consequence of the Phragmén–Lindelöf principle). On the other hand

$\hat{\psi}(s)$ decays faster than any polynomial, as explained in Lecture 1. Therefore, we can shift the contour on integration to $\Re(s) = 1/2$ and pick up poles at 1 and $1 \pm 2it$. We deduce that

$$I_2(t) = \text{Res}_{s=1} f(s,t) + \text{Res}_{s=1+2it} f(s,t) + \text{Res}_{s=1-2it} f(s,t) + \frac{1}{2\pi i}\int_{\Re(s)=1/2} f(s,t)ds.$$

We have

$$\text{Res}_{s=1\pm 2it} f(s,t) = \frac{8}{|\xi(1+2it)|^2}\hat{\psi}(1\pm 2it)\tilde{\gamma}(1\pm 2it,t)\frac{\zeta^2(1\pm 2it)\zeta(1\pm 4it)}{\zeta(2\pm 4it)}.$$

Again the polynomial growth of the $\tilde{\gamma}(1\pm 2it,t)$ in t, which follows from Stirling, and of $\zeta(1\pm it)$, together with the rapid decay of $\hat{\psi}(1\pm 2it)$ in t give that

$$\text{Res}_{s=1\pm 2it} f(s,t) \ll t^{-A}.$$

For the integral over $\Re(s) = 1/2$ we set $s = 1/2 + i\tau$ (to avoid confusion with t). We use Stirling again to see that

$$\tilde{\gamma}(1/2+i\tau,t) \ll |t^2 - \tau^2/4|^{-1/4} e^{-\pi(|t-\tau/2|+|t+\tau/2|)/2}.$$

We use the convexity estimate on $\zeta(1/2+i\tau)$ and the estimate on $\zeta(1+2i\tau)^{-1}$ from above to deduce that

$$Z(1/2+i\tau,t) \ll \tau^{1/2+\epsilon}\left|\zeta(1/2+i(\tau-2t))\right|\left|\zeta(1/2+i(\tau+2t))\right|.$$

We easily see the estimate (using Stirling)

$$\frac{1}{|\xi(1+2it)|^2} \ll \log^2 t\cdot e^{\pi t}.$$

Finally we can estimate

$$\begin{aligned}\int_{\Re(s)=1/2} f(s,t)ds \ll &\int_{-\infty}^{\infty}\frac{\log^2 t\cdot e^{\pi t}}{(1+|\tau|)^A}|t^2-\tau^2/4|^{-1/4}e^{-\pi(|t-\tau/2|+|t+\tau/2|)/2}\\ &\cdot\left|\zeta(1/2+i(\tau-2t))\right|\left|\zeta(1/2+i(\tau+2t))\right|d\tau\\ \ll &\frac{\log^2 t}{t^{1/2}}\int_{|\tau-2t|\le t^\epsilon}\frac{1}{(1+|\tau|)^A}\left|\zeta(1/2+i(\tau-2t))\right|^2+\left|\zeta(1/2+i(\tau+2t))\right|^2 d\tau\\ &+O(t^{-A})\\ &=O\left(\frac{\log^2 t}{t^{1/2}}\cdot\max_{|u-2t|<t^\epsilon}\left|\zeta(1/2+iu)\right|^2\right)+O(t^{-A}).\end{aligned}$$

□

Using the convexity bound on $\zeta(1/2+iu)$, i.e. $\zeta(1/2+iu) \ll u^{1/4+\epsilon}$ does not suffice to prove that the contribution of the contour integral over $\Re(s) = 1/2$ tends to 0. We need a result of the form $\zeta(1/2+iu) \ll u^{1/4-\delta}$ for some fixed δ. Such a result is called subconvexity bound (see section below for the L-function of a Maaß cusp form). In fact in the following Lecture 4 we will prove the Weyl bound

$$\zeta(1/2+it) \ll |t|^{1/6}\log t.$$

With the Weyl bound on $\zeta(1/2+it)$ we reduce the problem to calculating the residue at $s=1$.

2.4. **Evaluation of the residue at $s = 1$.** We write $\frac{|\xi(1+2it)|^2}{8} f(s,t) = B(s) = \zeta^2(s)G(s)$ with $G(s)$ holomorphic at $s=1$. The Laurent expansion of $\zeta(s)$ close to 1 is

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

with γ the Euler–Mascheroni constant. By squaring we get

$$\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + O(1),$$

where the last term is holomorphic in a neighbourhood of $s=1$. We get

$$\begin{aligned}\mathrm{Res}_{s=1}B(s) =& \lim_{s\to 1}\frac{d}{ds}(s-1)^2\zeta^2(s)G(s)\\ =& \lim_{s\to 1}\frac{d}{ds}(1+2\gamma(s-1)+O((s-1)^2))G(s)\\ =& 2\gamma G(1)+G'(1) = G(1)(2\gamma + G'(1)/G(1)).\end{aligned}$$

With $G(s) = \hat{\psi}(s)Z(s,t)\tilde{\gamma}(s,t)/\zeta^2(s)$ we see that

$$G(1) = \hat{\psi}(1)\frac{|\zeta(1+2it)|^2}{\zeta(2)}\frac{\Gamma^2(1/2)|\Gamma(1/2+it)|^2}{8\pi\Gamma(1)}.$$

We take into account the factor $\frac{8}{2\pi i|\xi(1+2it)|^2}$ to get

$$\begin{aligned}\frac{8}{|\xi(1+2it)|^2}G(1) =& \int_0^\infty \psi(y)\frac{dy}{y^2}\cdot\frac{8}{\pi^{-1}|\zeta(1+2it)|^2|\Gamma(1/2+it)|^2}\\ &\cdot\frac{|\zeta(1+2it)|^2}{\zeta(2)}\frac{\Gamma^2(1/2)|\Gamma(1/2+it)|^2}{8\pi\Gamma(1)}\\ =& \frac{6}{\pi}\int_0^\infty \psi(y)\frac{dy}{y^2}.\end{aligned}$$

Notice that we used the value or $\zeta(2)$, Mellin inversion to recover $\hat{\psi}(1)$, and the special value $\Gamma(1/2)=\sqrt{\pi}$. To compute $G'(1)/G(1)$ we use logarithmic differentiation to get

$$\begin{aligned}\frac{G'(1)}{G(1)} =& \frac{\hat{\psi}'(1)}{\hat{\psi}(1)} + \frac{\zeta'(1+2it)}{\zeta(1+2it)} + \frac{\zeta'(1-2it)}{\zeta(1-2it)} - \frac{\zeta'(2)}{\zeta(2)}\\ &+\frac{\Gamma'(1/2)}{\Gamma(1/2)} - \frac{\Gamma'(1)}{\Gamma(1)} - \log\pi + \frac{\Gamma'(1/2-it)}{2\Gamma(1/2-it)} + \frac{\Gamma'(1/2+it)}{2\Gamma(1/2+it)}.\end{aligned}$$

We need to study the behaviour of the above expression as $t \to \infty$. Ignoring the constants we concentrate of the logarithmic derivative of $\Gamma(s)$ and $\zeta(s)$. In Problem sheet 1 we saw as a consequence of Stirling's formula that

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-it\right)=\log\left(\frac{1}{4}+t^2\right)+O(t^{-1})=2\log t+O(t^{-1}).$$

The behaviour of $\zeta'(1+it)/\zeta(1+it)$ is trickier. It is relatively easy to prove

$$\frac{\zeta'(1+it)}{\zeta(1+it)} \ll \log^9 t,$$

see [11, 3.6.6], which is way too big. Using the zero-free region of Hadamard–de la Vallée Poussin, one can show, see [11, 3.11.9]

$$\frac{\zeta'(1+it)}{\zeta(1+it)} \ll \log t.$$

Unfortunately this does not suffice to find asymptotics in QUE for incomplete Eisenstein series. One needs the Weyl–Hadamard–de la Vallée Poussin method to deduce

$$\frac{\zeta'(1+it)}{\zeta(1+it)} \ll \frac{\log t}{\log\log t}.$$

For a reference look at [11, 5.17.4]. We will not prove it here. This concludes the proof of Proposition 2.2.

To complete the proof of Theorem 2.2 we need an approximation argument. We know that Maaß cusp forms and incomplete Eisenstein series are dense in the space of continuous functions vanishing at the cusp. For $F \in C_0(\Gamma\backslash\mathbb{H})$ and $\epsilon > 0$ we find

$$G = G_1 + G_2,$$

where G_1 is a finite sum of cusp forms, G_2 in the space of incomplete Eisenstein series i.e. $G_2(z) = E(h, z)$, and

$$\|G - F\|_\infty < \epsilon.$$

With $H = G - F$ rapidly decaying at the cusp, we can find $h_1 \geq 0$, rapidly decaying such that

$$E(h_1, z) \geq |H(z)|, \quad \int_{\Gamma\backslash\mathbb{H}} E(h_1, z)d\mu(z) < 10\epsilon.$$

Since the measure $d\mu_t$ are positive we see that

$$\limsup_{t\to\infty} \frac{1}{\log t}\left|\int_{\Gamma\backslash\mathbb{H}} H(z)d\mu_t(z)\right| \leq \frac{1}{\log t}\left|\int_{\Gamma\backslash\mathbb{H}} E(h_1, z)d\mu_t(z)\right| \leq \frac{30}{\pi}\epsilon.$$

3. Subconvexity

We saw in Lecture 2 that the completed L-function of u_j is invariant under $s \to 1-s$ and its Gamma factors are $\Gamma((s+it_j)/2)\Gamma((s-it_j)/2)$. We apply the Phragmén–Lindelöf principle on the vertical strip $\Re(s) \in [-\epsilon, 1+\epsilon]$. We use Stirling's formula to determine the behaviour of $L(u_j, s)$ on $\Re(s) = -\epsilon$. Writing

$$L(u_j, s) = \pi^{2s-1} \frac{\Gamma((1-s+it_j)/2)\Gamma((1-s-it-j)/2)}{\Gamma((s+it_j)/2)\Gamma((s-it_j)/2)} L(u_j, 1-s)$$

and using (2.1), and the fact that $L(u_j, s)$ is bounded by $L(u_j, \sigma)$ for $\Re(s) = \sigma = 1+\epsilon$, we get that for $\Re(s) = -\epsilon$

$$L(u_j, -\epsilon + it) \ll_{\epsilon, t_j} (1+|t|)^{1+\epsilon}.$$

Interpolating between the two lines $\Re(s) = -\epsilon$ and $\Re(s) = 1+\epsilon$, we get

$$L(u_j, 1/2 + it) \ll_{\epsilon, t_j} (1+|t|)^{1/2+\epsilon}.$$

This is the convexity bound. By the analysis in the previous lecture, we know that this does not suffice to show the vanishing of the contribution of u_j to the limit quantum measure of the Eisenstein series. However, we noticed that any bound of the type

$$L(u_j, 1/2 + it) \ll_{\epsilon, t_j} (1+|t|)^{1/2-\delta}$$

for $\delta > 0$ suffices. Such bounds are called subconvex bounds for $L(u_j, 1/2+it)$. The following is a result of T. Meurman [8]:

Theorem 3.1. *We have*

$$L(u_j, 1/2 + it) \ll_{\epsilon} t_j^{1/2} e^{\pi t_j/2} (1+|t|)^{1/3+\epsilon}.$$

We will not prove this theorem here.

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