QUANTUM ERGODICITY AND L-FUNCTIONS
LECTURE 2
FOURIER COEFFICIENTS OF MAASS FORMS

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1. Hecke operators

We notice that the fundamental domain of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is symmetric around the Imaginary axis. The symmetry can be described by the transformation $T_0(x+iy) = -x+iy$ or in complex notation $T_0(z) = -\bar{z}$. This involution ($T_0^2 = \mathrm{Id}$) reverses orientation but otherwise it preserves hyperbolic distance. It has two eigenvalues ± 1. Eigenfunctions with eigenvalue 1 are called even functions, and with eigenvalue -1 odd. It commutes with the Laplace operator, as this contains two derivatives in x, so when applying the chain rule twice one get two minus signs multiplied.

There are an infinite set of other operators T_n, $n = 1, 2, \dots$ called Hecke operators (see [2, p.116] and [1, p.41-49]) that are fundamental in the arithmetic theory. They are defined on automorphic function on $\Gamma\backslash\mathbb{H}$ by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \bmod (d)} f\left(\frac{az+b}{d}\right).$$

It is a fact that $T_n f$ is automorphic for Γ. It is an averaging operator, i.e. it samples the function at certain translates of z by matrices with integer entries and determinant n.

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{r \in \Gamma\backslash\Gamma_n} f(rz),$$

where Γ_n is the set of integer matrices of determinant n, $\Gamma\backslash\Gamma_n$ is the orbits of the action of Γ on Γ_n by left multiplication. The linear transformations $(az+b)/d$ commute with Δ, so $T_n\Delta = \Delta T_n$. The Hecke operators T_n commute with each other due to the relation

$$T_n T_m = \sum_{d|(m,n)} T_{mn/d^2}. \tag{1.1}$$

They are also self-adjoint with respect to the L^2 inner product: $\langle T_n f, g\rangle = \langle f, T_n g\rangle$. Therefore, the Laplace operator Δ and Hecke operators T_n, $n = 0, 1, \dots$ (including the involution T_0) can be simultaneously diagonalised. We will take from now on u_j to be a Hecke basis, i.e. u_j will be an eigenfunction of Δ and all Hecke operators. We denote the Hecke eigenvalues by λ_j, i.e. $T_n u_j = \lambda_j(n) u_j$. It can be proved that the

nth Fourier coefficient of u_j is proportional to $\lambda_j(n)$ (see [1, p. 48] for the holomorphic case). If u_j is an even Maaß form, then we can write its Fourier decomposition as

$$u_j(z) = \rho_j(1) \sum_{n\geq 1} \lambda_j(n) y^{1/2} K_{it_j}(2\pi n y) \cos(2\pi n x),$$

where $\rho_j(1)$ is a normalisation constant so that $\|u_j\| = 1$. Here t_j is the spectral parameter, i.e. the Laplace eigenvalue of u_j is $1/4 + t_j^2$.

Remark. We note some special but most important cases of (1.1). For $(m, n) = 1$ we have

$$T_{mn} = T_m T_n$$

and

$$T_{p^k} = T_{p^{k-1}} T_p - T_{p^{k-2}}.$$

Because the eigenvalues of the system of commuting operators satisfy the same algebraic relations, we get

$$\lambda_j(mn) = \lambda_j(m)\lambda_j(n), \quad (m,n) = 1, \quad \lambda_j(p^k) = \lambda_j(p^{k-1})\lambda_j(p) - \lambda_j(p^{k-2}). \tag{1.2}$$

The first says that the $\lambda_j(n)$'s and Fourier coefficients of u_j are multiplicative (apart from the factor $\rho_j(1)$. The second provides a recursive relation for coefficients indexed by powers of the same prime.

2. The L-function of a Maass form

We can now introduce the L-function attached to u_j for even Hecke eigenfunctions. Modifications are necessary for u_j odd, see [1, p. 107]. We write dy/y, since this is an invariant differential under dilations. We set $L(u_j, s) = \sum_n \lambda_j(n) n^{-s}$. We consider the integral

$$\begin{aligned}\int_0^\infty u_j(iy) y^{s-1/2} \frac{dy}{y} &= \int_0^1 u_j(iy) y^{s-1/2} \frac{dy}{y} + \int_1^\infty u_j(iy) y^{s-1/2} \frac{dy}{y} \\ &= \int_1^\infty u_j(i/y) y^{1/2-s} \frac{dy}{y} + \int_1^\infty u_j(iy) y^{s-1/2} \frac{dy}{y} \\ &= \int_1^\infty u_j(iy)(y^{1/2-s} + y^{s-1/2}) \frac{dy}{y},\end{aligned}$$

where we changed variable $y \to y^{-1}$ and the automorphy relation $u_j(z) = u_j(-1/z)$ for $z = iy$. The original integral converges for $\Re(s) > 0$. The last one converges for all $s \in \mathbf{C}$. It provides the analytic continuation of the integral in the whole complex plane and we see the symmetry $s \to 1 - s$. We use the Fourier expansion of u_j to express the integral in terms of an L-function.

$$\begin{aligned}\int_0^\infty u_j(iy)y^{s-1/2}\frac{dy}{y} &= \rho_j(1)\sum_{n\geq 1}\lambda_j(n)\int_0^\infty K_{it_j}(2\pi ny)y^s\frac{dy}{y}\\ &= \rho_j(1)\sum_{n\geq 1}\lambda_j(n)\int_0^\infty K_{it_j}(y)\left(\frac{y}{2\pi n}\right)^s\frac{dy}{y}\\ &= \rho_j(1)\sum_{n\geq 1}\frac{\lambda_j(n)}{n^s}(2\pi)^{-s}2^{s-2}\Gamma\left(\frac{s+it_j}{2}\right)\Gamma\left(\frac{s-it_j}{2}\right)\\ &= \rho_j(1)2^{-2}\pi^{-s}\Gamma\left(\frac{s+it_j}{2}\right)\Gamma\left(\frac{s-it_j}{2}\right)L(u_j,s),\end{aligned}$$

where we changed variable $2\pi ny\to y$ and used the Mellin transform of the K-Bessel function (see [1, Lemma 1.9.1, p. 106] or [2, p.105])

$$\int_0^\infty K_\nu(y)y^s\frac{dy}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right),\quad \Re(s)>|\Re(\nu)|\,.$$

We get the theorem:

Theorem 2.1. *For u_j even the completed L-function*

$$\Lambda(s)=\pi^{-s}\Gamma\left(\frac{s+it_j}{2}\right)\Gamma\left(\frac{s-it_j}{2}\right)L(u_j,s)$$

has analytic continuation to the whole complex plane and satisfied the functional equation $\Lambda(1-s)=\Lambda(s)$.

By (1.2) we know that the L-function $L(f,s)$ can be factored into its Euler product:

$$L(u_j,s)=\prod_p\sum_{k=0}^\infty\frac{\lambda_j(p^k)}{p^{ks}}.$$

We now show that

$$\sum_{k=0}^\infty \lambda_j(p^k)X^k = \frac{1}{1-\lambda_j(p)X+X^2} \tag{2.1}$$

for X a formal variable. Applying this with $X=p^{-s}$ we get

$$L(u_j,s)=\prod_p\frac{1}{1-\lambda_j(p)p^{-s}+p^{-2s}}.$$

The equation (2.1) is equivalent with

$$1=(1-\lambda_j(p)X+X^2)\sum_{k=0}^\infty\lambda_j(p^k)X^k=\sum_{k=0}^\infty\left(\lambda_j(p^k)X^k-\lambda_j(p^k)\lambda_j(p)X^{k+1}+\lambda_j(p^k)X^{k+2}\right)$$

$$= \sum_{k=0}^{\infty} (\lambda_j(p^k) - \lambda_j(p^{k-1})\lambda_j(p) + \lambda_j(p^{k-2}))X^k,$$

which is equivalent to the second relation in (1.2).

2.1. **The Ramanujan–Petersson conjecture.** The quadratic polynomial $1-\lambda_j(p)X+X^2$ can be factored as $(1-\alpha_j(p)X)(1-\beta_j(p)X)$ with

$$\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1.$$

The size of $\alpha_j(p)$ and $\beta_j(p)$ has considerable meaning. The Ramanujan–Petersson conjecture for Maaß forms predicts that

$$|\alpha_j(p)| = |\beta_j(p)| = 1,$$

so that

$$\alpha_j(p) = \beta_j(p)^{-1} \quad \text{and} \, |\lambda_j(p)| \leq 2.$$

This conjecture is open, unlike the case of holomorphic cusp forms, which was settled by Deligne. The best result towards the Ramanujan–Petersson conjecture in due to Kim and Sarnak: $|\alpha_j(p)|, |\beta_j(p)| \leq p^{7/64}$ [3]. Using (1.2) one can prove inductively that

$$\lambda_j(p^k) = \sum_{0\leq l\leq k} \alpha_j(p)^l \beta_j(p)^{k-l}.$$

This means that the Ramanujan–Petersson conjecture for the Fourier coefficients of u_j is $\left|\lambda_j(p^k)\right| \leq d(p^k) = k+1$ with $d(n)$ the standard divisor function. For general n, using (1.2) we get the conjecture

$$|\lambda_j(n)| \leq d(n).$$

3. The Rankin–Selberg L-function

Let f and g be two automorphic forms, which are Hecke eigenfunctions with systems of Hecke eigenvalues $\lambda_j(n)$ and $\lambda_m(n)$. Tacitly we allow that g is not cuspidal, i.e. has a non-vanishing zero Fourier coefficient. This is important if we want to allow g to be an Eisenstein series. The Rankin–Selberg L-function is defined as the Dirichlet series (with multiplicative coefficients):

$$L(f \otimes g, s) = \zeta(2s) \sum_{n\geq 1} \frac{\lambda_j(n)\lambda_m(n)}{n^s}.$$

We analyse its Euler factors. From

$$L(f,s) = \prod_p (1-\alpha_j(p)p^{-s})^{-1}((1-\beta_j(p)p^{-s})^{-1}, \quad L(g,s) = \prod_p (1-\alpha_m(p)p^{-s})^{-1}((1-\beta_m(p)p^{-s})^{-1}$$

we deduce that

$$L(f\otimes g,s)=\prod_p(1-\alpha_j(p)\alpha_m(p)p^{-s})^{-1}(1-\alpha_j(p)\beta_m(p)p^{-s})^{-1}\\ \times(1-\beta_j(p)\alpha_m(p)p^{-s})^{-1}(1-\beta_j(p)\beta_m(p)p^{-s})^{-1},$$

using [1, Lemma 1.6.1] and that $\alpha_j(p)\beta_j(p)=1$ and $\alpha_m(p)\beta_m(p)=1$. We notice that $\prod_p(1-p^{-2s})^{-1}=\zeta(2s)$. It is clear that polynomial bounds on $\lambda_j(n)$ and $\lambda_m(n)$ give that the Rankin–Selberg L-function converges in an appropriate right half-plane. Its analytic continuation is provided by the method of integral representation. We start with the integral

$$I(s)=\int_{\Gamma\backslash\mathbb{H}}f(z)g(z)E(z,s)d\mu(z)$$

and unfold to get

$$I(s)=\int_{\Gamma_\infty\backslash\mathbb{H}}f(z)g(z)y^sd\mu(z)=\int_0^\infty\left(\int_0^1f(z)g(z)dx\right)y^{s-2}dy\\ =\int_0^\infty\rho_j(1)\rho_m(1)\frac{1}{2}\sum_{n\geq1}\lambda_j(n)\lambda_m(n)K_{it_j}(2\pi ny)K_{it_m}(2\pi ny)y^s\frac{dy}{y},$$

where we have used the orthogonality of $\cos(2\pi nx)$ to $\cos(2\pi kx)$ for $n\neq k$, while $\int_0^1\cos^2(2\pi nx)dx=1/2$. We now change variable $2\pi ny\to y$ and use the Mellin transform of the product of two K-Bessel functions (see [2, p.205]

$$\int_0^\infty K_\mu(y)K_\nu(y)y^{s-1}dy=2^{s-3}\Gamma(s)^{-1}\prod\Gamma\left(\frac{s\pm\mu\pm\nu}{2}\right)$$

for $\Re(s)>|\Re(\mu)|+|\Re(\nu)|$ and with all four choices of $\pm$ in the product. We get

$$I(s)=\rho_j(1)\rho_m(1)\frac{1}{2}\sum_{n\geq1}\lambda_j(n)\lambda_m(n)(2\pi n)^{-s}\int_0^\infty K_{it_j}(y)K_{it_m}(y)y^s\frac{dy}{y}\\ =\rho_j(1)\rho_m(1)\frac{1}{2}2^{-3}\pi^{-s}\frac{1}{\Gamma(s)\zeta(2s)}\prod\Gamma\left(\frac{s\pm it_j\pm it_m}{2}\right)L(f\otimes g,s).$$

The integral $I(s)$ has meromorphic continuation in the whole complex plane, by the analytic continuation of $E(z,s)$. The simple pole at $s=1$ of $E(z,s)$ gives at most a simple pole at $s=1$ with residue $3\pi^{-1}\langle f,g\rangle$. The constant is an easy explicit calculation using $\phi(s)$. If now f is perpendicular to g because they have different eigenvalues for Δ, the integral $I(s)$ is holomorphic at $s=1$. The Gamma factors in front of $L(f\otimes g,s)$ are regular and nonzero close to $s=1$, so $L(f\otimes g,s)$ is regular at $s=1$. On the other hand, if $f=g$, then we get $\mathrm{res}(I(s),1)=3\pi^{-1}$. We conclude that in this case $L(f\otimes g,s)$ has a pole at $s=1$.

The functional equation of $L(f\otimes g, s)$ can be obtained as follows: $I(s) = \phi(s)I(1-s)$ with $\phi(s) = \xi(2-2s)/\xi(2s)$. Therefore, we have

$$\pi^{-s}\Gamma(s)\zeta(2s)I(s) = \pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)I(1-s).$$

We can ignore the constants in the integral representation and see that

$$\Lambda(s) = \pi^{-2s}\prod\Gamma\left(\frac{s\pm it_j\pm it_m}{2}\right)L(f\otimes g, s)$$

satisfies $\Lambda(s) = \Lambda(1-s)$. We repeat that we have assumed that both f and g are even.

References

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