QUANTUM ERGODICITY AND SUBCONVEXITY OF L-FUNCTIONS
LECTURE 1
HARMONIC ANALYSIS ON THE HYPERBOLIC PLANE

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1. Review of classical Fourier analysis

Classical Fourier analysis of periodic functions with period e.g. one uses the standard exponentials

$$e(nx), \quad n \in \mathbb{Z}, \quad \text{with} \quad e(x) := e^{2\pi i x}.$$

Let f be a periodic function with period 1, i.e. $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Such a function can be considered as a function on the quotient space $S^1 = \mathbb{R}/\mathbb{Z}$. Let $f \in L^1(\mathbb{R}/\mathbb{Z})$. We define its Fourier coefficients by

$$\hat{f}(n) = \int_0^1 f(x) e(-nx) dx, \quad n \in \mathbb{Z}, \tag{1.1}$$

where, by periodicity, we could have used any interval of length 1 for the integration. This is spectral analysis. Spectral synthesis is interested in the inversion of this:

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e(nx). \tag{1.2}$$

It is known that this result does not hold pointwise for all $f \in L^1(\mathbb{R}/\mathbb{Z})$. However, it is true in the L^2 sense: Let $f \in L^2(\mathbb{R}/\mathbb{Z})$. The Fourier series on the right of (1.2) converges in the L^2 sense to the function f. This means that the partial sums

$$f_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e(nx)$$

satisfy:

$$\|f_N - f\|_2 \to 0, \quad N \to \infty.$$

In the language of Hilbert spaces, the set of functions $e(nx)$, $n \in \mathbb{Z}$ is an orthonormal basis for the space $L^2(\mathbb{R}/\mathbb{Z})$ of periodic square-integrable functions on $[0, 1]$. This space is actually a Hilbert space with an inner product given for two elements f and g by

$$\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} \, dx.$$

The basic Fourier series result for Hilbert spaces can be written as

$$f = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle\, e_n,$$

where e_n is an orthonormal basis. In fact Plancherel's theorem says that

$$\|f\|^2 = \sum_n |\langle f, e_n\rangle|^2,$$

which translates into the usual Parceval identity

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2.$$

Going back to pointwise convergence, it is known that if the Fourier series in (1.2) is absolutely convergent, then it converges to f pointwise. It follows that if f is continuously differentiable on $\mathbb{R}/\mathbb{Z}$, then pointwise convergence holds.

It is worth noting that $e(nx)$ satisfy the differential equation $f'' + \lambda f = 0$ with $\lambda = 4\pi^2 n^2$. We call the operator d^2/dx^2 the Laplace operator on $\mathbb{R}/\mathbb{Z}$. So classical Fourier analysis on the circle amounts to finding an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$ consisting of eigenfunctions of the Laplace operator. We remark that this basis is countable and, in fact, the eigenvalue parameters $4\pi^2n^2$ are a discrete set in $[0,\infty)$.

The situation changes drastically if we drop periodicity and want to do Fourier analysis on $\mathbb{R}$. One defines the Fourier transform of $f \in L^1(\mathbb{R})$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e(-\xi x)dx, \quad \xi \in \mathbb{R}, \tag{1.3}$$

and Fourier inversion now takes the form of an integral (inverse Fourier transform)

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e(\xi x)\, d\xi. \tag{1.4}$$

For later purposes it is worth mentioning that $\hat{f}(0) = \int_{\mathbb{R}} f$. We notice here that the exponentials $e(nx)$ are not even functions in $L^2(\mathbb{R})$, as they have modulus 1. Nevertheless this inversion formula is true for a dense subset of $L^2(\mathbb{R})$, in fact on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. For such functions the Fourier transform is defined pointwise. Here the Plancherel theorem takes the form

$$\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2\, d\xi.$$

We can extend the transform to all of $L^2(\mathbb{R})$ as a unitary operator by continuity.

We notice that the exponentials $e(\xi x)$ are also eigenfunctions of the differential operator d^2/dx^2 with eigenvalue $4\pi^2|\xi|^2$, even if they are not in the Hilbert space $L^2(\mathbb{R})$. However, they are 'close' to be in this space in the following sense. First of all they are bounded functions. Second, we can solve the differential equation $d^2f/dx^2+\lambda f = 0$, and we easily see that with $\lambda = 4\pi^2\xi^2$ the solutions are $f(x) = e(\xi x)$

even for *complex* ξ. We check that if $\Im\xi \neq 0$, then $e(\xi x)$ grows exponentially either at $+\infty$ or at $-\infty$. Such functions clearly are not in $L^2(\mathbb{R})$. When $\Im\xi = 0$, we get the standard exponentials used in Fourier analysis, for which $\int_{-T}^{T} |e(\xi x)|^2\, dx = 2T$. Moreover, we have a continuous set of spectral values: $4\pi^2 |\xi|^2$, as ξ varies in the reals. We notice that ξ and $-\xi$ provide the same eigenvalue, i.e. any linear combination of $e(\xi x)$ and $e(-\xi x)$ is an eigenfunction.

2. Description of the modular surface and its spectral decomposition

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ the group of two by two matrices with integer entries and determinant 1. This acts on the hyperbolic plane $\mathbb{H}$ by linear fractional transformations. The fundamental domain can be taken to be the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$:

$$D = \{z \in \mathbb{H}, -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}, |z| \geq 1\}.$$

We denote it by $\Gamma\backslash\mathbb{H}$. For background material on the hyperbolic plane $\mathbb{H}$ and its geometry, look at the notes [2].

We are interested in functions on $\Gamma\backslash\mathbb{H}$, i.e. functions $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = f(z), \quad \forall \gamma \in \Gamma, \quad z \in \mathbb{H}.$$

We are interested in the harmonic analysis on the quotient $\Gamma\backslash\mathbb{H}$, which is a locally symmetric space. We work with $L^2(\Gamma\backslash\mathbb{H})$, which is defined using the usual L^2 inner product

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}\, d\mu(z).$$

Recall from hyperbolic geometry that $d\mu(z) = dxdy/y^2$ and that the area of $\Gamma\backslash\mathbb{H}$ is $\pi/3$. Unfortunately for $\Gamma\backslash\mathbb{H}$ we cannot be as explicit as we were with the Fourier analysis on $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}$. The Laplace operator is of fundamental importance and is defined by:

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{2.1}$$

We would like to work with the automorphic Laplacian, i.e. the self-adjoint (extension) of the differential operator Δ on automorphic functions. Its spectral analysis on $L^2(\Gamma\backslash\mathbb{H})$ and, consequently, spectral synthesis, is much more complicated and exhibits phenomena that appeared in both $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}$. There exists a discrete part of the spectrum of the Laplacian given by L^2-eigenfunctions u_j, called Maaß forms, satisfying $\Delta u_j + \lambda_j u_j(z) = 0$ and an (absolutely) continuous part of the spectrum covering the interval $[1/4, \infty)$ once (since Γ has one cusp) provided by non-holomorphic

Eisenstein series $E(z, 1/2 + it)$ satisfying the eigenvalue equation

$$\Delta E(z, 1/2 + it) + \left(\frac{1}{4} + t^2\right) E(z, 1/2 + it) = 0, \quad t \in \mathbb{R}.$$

The discrete part is similar to what happens on $\mathbb{R}/\mathbb{Z}$ and the continuous similar to $\mathbb{R}$. Both are necessary to recover Parceval's identity (completeness of the spectral decomposition).

Since Γ is generated by the translation $T(z) = z+1$ and the inversion $S(z) = -1/z$ and the first maps the left vertical side of the fundamental domain to the right side, while S maps the right arc $\{z : |z| = 1, x > 0\}$ to the left arc $\{z : |z| = 1, x < 0\}$, we can consider that the automorphy condition for the Laplace operator corresponds to the boundary conditions that f has the same values on corresponding points of the vertical rays and points on the arc.

There is a simple $L^2(\Gamma\backslash\mathbb{H})$-eigenvalue: 0, since any constant function satisfies the eigenvalue equation with $\lambda_0 = 0$. Moreover, constants are clearly Γ-invariant. We normalize this eigenfunction to have L^2-norm one, which is equivalent to taking $u_0(z) = \mathrm{vol}(\Gamma\backslash\mathbb{H})^{-1/2}$. Let $\mathcal{B}(\Gamma\backslash\mathbb{H})$ be the smooth and bounded automorphic functions. We first look at the subspace of $\mathcal{B}(\Gamma\backslash\mathbb{H}) \subset L^2(\Gamma\backslash\mathbb{H})$ consisting of cuspidal functions:

$$\mathcal{C}(\Gamma\backslash\mathbb{H}) = \{f \in L^2(\Gamma\backslash\mathbb{H}) | f \text{ smooth, bounded}, f_0(y) = 0\}. \tag{2.2}$$

Here $f_0(y)$ is the zero-th Fourier coefficient of f.

Remark. Clearly Δ maps $\mathcal{C}(\Gamma\backslash\mathbb{H})$ into itself. Therefore, the same is true for its orthogonal complement.

Theorem 2.1. *The automorphic Laplace operator Δ has pure point spectrum on $\mathcal{C}(\Gamma\backslash\mathbb{H})$ i.e. this space is spanned by cuspidal Maaß forms, which we call cusp forms or Maaß cusp forms. For a complete orthonormal system of cusp forms $u_j(z)$, $j = 1, \dots$ and every $f \in \mathcal{C}(\Gamma\backslash\mathbb{H})$ we have the expansion*

$$f(z) = \sum_{j=1}^{\infty} \langle f, u_j \rangle u_j(z).$$

This expansion converges in the norm topology. If $f \in \mathcal{B}(\Gamma\backslash\mathbb{H})$ has also $\Delta f \in \mathcal{B}(\Gamma\backslash\mathbb{H})$, then the series converges absolutely and uniformly on compact sets.

The fact that $C(\Gamma\backslash\mathbb{H})$ is infinite dimensional is by no means obvious.

The main result in the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$ is:

Theorem 2.2. [1, Th. 4.7, Th. 7.3] *Every $f \in L^2(\Gamma\backslash\mathbb{H})$ has the expansion*

$$f(z) = \sum_{j=0}^{\infty} \langle f, u_j \rangle u_j(z) + \frac{1}{4\pi} \int_{\mathbb{R}} \langle f, E(\cdot, 1/2 + it) \rangle E(z, 1/2 + it)\, dt. \tag{2.3}$$

The convergence holds in the norm topology, and, if, additionally, f and Δf are smooth and bounded, then the expansion (2.3) *converges pointwise, absolutely and uniformly on compact sets of* $\Gamma\backslash\mathbb{H}$.

For the definition and properties of Eisenstein series, see the discussion below.

We also need the space $\mathcal{E}(\Gamma\backslash\mathbb{H})$ of incomplete Eisenstein series. Let ψ be a compactly supported function on $(0,\infty)$ (more generally one can take functions which are rapidly decaying at 0 and infinity: i.e. for every positive integer N, we have $\psi(y) = O_N(y^N)$, as $y \to 0$ and $\psi(y) = O_N(y^{-N})$, as $y \to \infty$. We define the incomplete Eisenstein series (occasionally but wrongly called incomplete theta series)

$$E(\psi, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\Im(\gamma z)).$$

For ψ compactly supported on $(0,\infty)$, $E(\psi, z)$ is bounded and automorphic.

For $\psi(y) = y^s$ we get the Eisenstein series (notice that the conditions for ψ are not satisfied):

$$E(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s. \tag{2.4}$$

Here $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and Γ_∞ is the cyclic subgroup generated by $T : z \mapsto z + 1$. This series converges absolutely and locally uniformly for $\sigma = \Re(s) > 1$. The Eisenstein series $E(z, s)$ admits a Fourier expansion of the cusp $i\infty$, see e.g. [1, (3.25)]

$$\begin{aligned} E(z, s) &= \sum_{n\in\mathbb{Z}} a_n(y, s)e^{2\pi inx} \\ &= y^s + \phi(s)y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{n\neq 0} |n|^{s-1/2} \sigma_{1-2s}(|n|)K_{s-1/2}(2\pi |n| y)e^{2\pi inx}. \end{aligned} \tag{2.5}$$

Here $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function satisfying the functional equation $\xi(s) = \xi(1-s)$, $\sigma_c(n)$ is the sum of the cth powers of the divisors of n, and $K_s(y)$ is the K-Bessel function. The scattering matrix is

$$\phi(s) = \frac{\xi(2-2s)}{\xi(2s)}. \tag{2.6}$$

The Fourier series converges for $z \in \mathbb{H}$ due to the rapid decay of the K-Bessel functions, see [1, B.36]. Therefore, $E(z, s)$ is a holomorphic function of s away from the poles of $\phi(s)$ and zeros of $\xi(2s)$. Zeros of $\xi(2s)$ occur at $\rho/2$, where ρ are the non-trivial zeros of the Riemann zeta function. So RH is equivalent with the statement that the poles of $E(z, s)$ in $\Re(s) < 1/2$ have real part 1/4. It is doubtful this will help with proving RH. But it is a spectral interpretation of the Riemann zeros. Writing $\xi(2-2s) = \xi(2s-1)$ we also see that the zero Fourier coefficient of $E(z, s)$ has a pole at $s = 1$ corresponding to the pole of $\zeta(s)$ at $s = 1$. Notice that if $E(z, s)$ is holomorphic at a point s, then all its Fourier coefficients are as well. This provides the meromorphic continuation of $E(z,s)$ in the whole complex plane.

The Eisenstein series satisfies the functional equation $E(z,s)=\phi(s)E(z,1-s)$. This is seen by matching the Fourier coefficients of $\phi(s)E(z,1-s)$ with those of $E(z,s)$. There are only two observations needed: for each natural number a we have

$$a^{1/2-s}\sigma_{2s-1}(a)=a^{s-1/2}\sigma_{1-2\sigma}(a)$$

and

$$K_\nu(y)=\int_0^\infty e^{-y\cosh t}\cosh(\nu t)dt=K_{-\nu}(y),$$

see [1, p.205].

The incomplete Eisenstein series rarely is an eigenfunction of Δ. It is important that we express it as a contour integral of the Eisenstein series

$$E(\psi,z)=\frac{1}{2\pi i}\int_{(\sigma)}E(z,s)\hat{\psi}(s)\,ds,$$

where $\sigma>1$ and

$$\hat{\psi}(s)=\int_0^\infty \psi(y)y^{-s-1}dy$$

is the Mellin transform of ψ. This is an easy application of the inversion of Mellin transform:

$$\psi(y)=\frac{1}{2\pi i}\int_{(\sigma)}\hat{\psi}(s)y^s ds.$$

All we have to do is plug $y=\Im(\gamma z)$ and sum over $\gamma\in\Gamma_\infty\backslash\Gamma$. The interchange of the summation and integration is obvious, since for $\sigma>1$ the series for $E(z,s)$ converges absolutely. We also need to notice that the assumptions on ψ imply by repeated integration by parts that $\hat{\psi}(s)=O_A((1+|s|)^{-A})$ for all $A>0$.

Remark. This is not the standard normalisation of Mellin transform (usually one uses $+s$ in the transform and $-s$ in the inverse.

Theorem 2.3. *The orthogonal complement of $\mathcal{E}(\Gamma\backslash\mathbb{H})$ in $L^2(\Gamma\backslash\mathbb{H})$ is the closure of $\mathcal{C}(\Gamma\backslash\mathbb{H})$ in $L^2(\Gamma\backslash\mathbb{H})$. This gives the decomposition*

$$L^2(\Gamma\backslash\mathbb{H})=\overline{\mathcal{C}}(\Gamma\backslash\mathbb{H})\oplus\overline{\mathcal{E}}(\Gamma\backslash\mathbb{H}),$$

where overline denotes the closure in the Hilbert space $L^2(\Gamma\backslash\mathbb{H})$.

Proof. Let f be automorphic and integrable over $\Gamma\backslash\mathbb{H}$. Then $f(x+1+iy)=f(x+iy)$, which allows to expand f in Fourier series in x:

$$f(z)=f_0(y)+\sum_{n\neq 0}f_n(y)e(nx).$$

Let us assume moreover that f is perpendicular to $\mathcal{E}(\Gamma\backslash\mathbb{H})$. Let $E(\psi, z)$ be an incomplete Eisenstein series. Then we get

$$0 = \langle f, E(\psi, z)\rangle = \langle f, \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\Im(\gamma z))\rangle = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\Im(\gamma z)}\, d\mu(z).$$

We unfold (à la Rankin–Selberg): setting $z' = \gamma z$ we change variables, noticing that the hyperbolic measure is invariant for $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, and observe that as γ runs over the cosets $\Gamma_\infty\backslash\Gamma$ (and we take appropriate representatives of the cosets) the sets $\gamma^{-1}D$ cover the strip $\{z \in \mathbb{H} : 0 < x < 1\}$, which is the fundamental domain of the infinite cyclic group Γ_∞ to get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma^{-1}\Gamma\backslash\mathbb{H}} f(z)\overline{\psi(\Im z)}d\mu(z) = \int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{\psi(\Im z)}d\mu(z)$$

$$= \int_0^\infty \left(\int_0^1 f(z)dx\right)\overline{\psi(y)}y^{-2}dy = \int_0^\infty f_0(y)\overline{\psi(y)}y^{-2}dy.$$

This implies that $f_0(y) = 0$ i.e. $f \in \mathcal{C}(\Gamma\backslash\mathbb{H})$. □

Remark. The unfolding à la Rankin–Selberg appear often. It concerns the inner product of an automorphic form, here $f(z)$, and a series over $\Gamma_\infty\backslash\Gamma$. It is also a general fact that if $\Gamma_1 \subset \Gamma_2$ are subgroups, and D_2 is a fundamental domain of Γ_2, then a fundamental domain of Γ_1 can be taken to be $D_1 = \cup_{\gamma\in\Gamma_1\backslash\Gamma_2} gD_2$.

The spectral analysis on the continuous spectrum uses a transform, called the Eisenstein transform. It intertwines the Laplace operator on a subspace of $\mathcal{E}(\Gamma\backslash\mathbb{H})$ with a multiplication operator M on $C_0^\infty(\mathbb{R}^+)$. The Eisenstein transform maps functions in $C_0^\infty(\mathbb{R}^+)$ to $L^2(\Gamma\backslash\mathbb{H})$ by

$$E(f)(z) = \frac{1}{4\pi}\int_0^\infty f(r)E(z, 1/2 + ir)dr.$$

This is an isometric map, if we equip $L^2(\mathbb{R}^+)$ with the inner product $\langle f, g\rangle = (2\pi)^{-1}\int_0^\infty f(r)\overline{g(r)}dr$. The fact that this is an isometry is not obvious at all! For a proof look at [1, Prop. 7.1], where the Maaß–Selberg relations are used, which give the asymptotic inner-product of Eisenstein series with itself. The image of the map is called $\mathcal{E}_\infty(\Gamma\backslash\mathbb{H})$.

If we define $Mf(r) = -(r^2 + 1/4)f(r)$, then $\Delta E(f) = E(Mf)$. The spectral theorem in the continuous spectrum can now be stated as follows:

Theorem 2.4. *We have the orthogonal splitting*

$$\mathcal{E}(\Gamma\backslash\mathbb{H}) = \mathcal{R}(\Gamma\backslash\mathbb{H}) \oplus \mathcal{E}_\infty(\Gamma\backslash\mathbb{H}),$$

where $\mathcal{R}(\Gamma\backslash\mathbb{H})$ is the one dimensional space of constants. The spectrum of Δ on $\mathcal{E}_\infty(\Gamma\backslash\mathbb{H})$ is absolutely continuous and covers the interval $[1/4, \infty)$ with multiplicity 1.

Remark. The space $\mathcal{R}_\infty(\Gamma\backslash\mathbb{H})$ is the space of residues of Eisenstein series in the interval $(1/2, 1]$. Here there is only one: the constant function. In general this is a finite dimensional space. For the modular group Γ we can see that there is no other pole of $E(z,s)$ in the interval $(1/2,1]$, since $\xi(2s-1)$ has only pole at $s=1$ and the denominator $\xi(2s)$ of $\phi(s)$ has no zeros in this interval, due to the non-vanishing of $\zeta(s)$ in the domain of absolute convergence $\sigma > 1$. This is due to the Euler product. It is also known and can be proved using the Rayleigh quotient that there are no Maaß cusp forms with eigenvalue less than 1/4 for this group. Selberg's eigenvalue conjecture says that $\lambda_1 \geq 1/4$ for all congruence groups.

Remark. It is customary in the field to write the eigenvalue equation as $\Delta u + s(1-s)u = 0$. The map $\lambda = s(1-s)$ is creating a double cover of the λ-plane. The 'physical plane' λ is opened up along the ray $[0,\infty)$, where the spectrum of $-\Delta$ should be and corresponds to the right half-plane $\Re(s) \geq 1/2$. The left half-plane $\Re(s) < 1/2$ is where analytic continuations have to be worked out and poles of $E(z,s)$ in it are called scattering poles or resonances.

With this normalisation $s_0 = 1$ corresponds to $\lambda_0 = 0$. The continuous spectrum $[1/4,\infty)$ corresponds to the critical line $\Re(s) = 1/2$. The small eigenvalues $\lambda_j < 1/4$ correspond to real $s_j \in (1/2,1]$. The Maaß cusp forms for the modular group correspond to points s_j and their conjugate on the critical line. They are mysterious and may remind us of Riemann zeros. There are two fundamental differences: RH is not known to be true, while it is known for the spectral parameters s_j that $\Re(s_j) = 1/2$. Moreover, unlike the counting of Riemann zeros (von Mangoldt formula), Weyl's law (hard to prove) says that

$$N(T) = |\{s_j : 0 < \Im(s_j) \leq T\}| \sim \frac{\text{Area}(\Gamma\backslash\mathbb{H})}{4\pi} T^2.$$

References

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