2. Lecture 2: (Metric) Diophantine approximation

Problem 8. *Minkowski's Convex Body Theorem.* Prove that if C is a closed symmetric[6] convex body in $\mathbb{R}^d$ with $\mathrm{vol}(C) \geq 2^d$, then there exists a non-zero vector in $\mathbb{Z}^d \cap C$.

Hint: First assume $\mathrm{vol}(C) > 2^d$. Then an asymptotic packing/volume argument implies that the translated bodies $\boldsymbol{m} + \frac{1}{2}C$ with $\boldsymbol{m}$ running through $\mathbb{Z}^d$ *cannot* be pairwise disjoint; i.e. there exist $\boldsymbol{m} \neq \boldsymbol{m}'$ in $\mathbb{Z}^d$ such that $(\boldsymbol{m} + \frac{1}{2}C) \cap (\boldsymbol{m}' + \frac{1}{2}C) \neq \emptyset$. This is easily seen to imply the existence of a non-zero vector in $\mathbb{Z}^d \cap C$. Finally if $\mathrm{vol}(C) \geq 2^d$, apply the previous result to the body tC for $t > 1$, and let $t \to 1$.

Problem 9. For $d = 1$, prove that $x \in \mathbb{R}$ is singular if and only if x is rational.

Problem 10. (a) Prove the following more general version of Dirichlet's Theorem ("DT for m linear forms in n variables"): For every $X \in M_{n,m}(\mathbb{R})$ and $Q > 1$, there exist $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{\boldsymbol{0}\})$ satisfying

$$\|\boldsymbol{q}X - \boldsymbol{p}\| \leq Q^{-\frac{1}{m}} \qquad \text{and} \qquad \|\boldsymbol{q}\| \leq Q^{\frac{1}{n}}.$$

(b) From the result in (a), give a natural definition of what it means for $X \in M_{n,m}(\mathbb{R})$ to be *badly approximable* and *singular*, respectively. Now prove that if we write

$$u(X) := \begin{pmatrix} I_m & 0 \\ X & I_n \end{pmatrix} \qquad \text{and} \qquad L_X := \mathbb{Z}^{m+n} u(X) \in X_{m+n}$$

and consider the following diagonal flow on X_{m+n}:

$$a_t = \begin{pmatrix} e^{nt} I_m & 0 \\ 0 & e^{-mt} I_n \end{pmatrix} \qquad (t \in \mathbb{R}),$$

then we have: $X \in M_{n,m}(\mathbb{R})$ is badly approximable iff the orbit $\{L_X a_t\}_{t \geq 0}$ in X_{m+n} is bounded. $X \in M_{n,m}(\mathbb{R})$ is singular iff the orbit $\{L_X a_t\}_{t \geq 0}$ in X_{m+n} is divergent.

[6]symmetric means: $C = -C$.