PROBLEMS (HOMOGENEOUS DYNAMICS AND NUMBER THEORY)

1. Problems for Lecture 1 (Diagonal and unipotent flows)

Problem 1. *Identification of $PSL_2(\mathbb{R})$ and the unit tangent bundle of $\mathbf{H}$;* [7, pp. 8–9, Exercises 10–11].

Let $\mathbf{H} = \{x + iy \in \mathbb{C} \ : \ y > 0\}$ be the *hyperbolic upper half plane* (or simply *hyperbolic plane*), with Riemannian metric $\langle v, w\rangle := y^{-1} v \cdot w$ for any $v, w \in T_{x+iy}(\mathbf{H})$, where $v \cdot w$ is the usual Euclidean inner product on $\mathbb{R}^2 \cong \mathbb{C}$.

(a) Show that the formula

$$g(z) = \frac{az+b}{cz+d} \qquad \text{for } \ z \in \mathbf{H} \text{ and } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

defines an action of $\mathrm{SL}_2(\mathbb{R})$ by isometries on $\mathbf{H}$, and show that this action is transitive.

(b) The *unit tangent bundle* $T^1\mathbf{H}$ consists of the tangent vectors of length 1. By differentiation we obtain an action of $\mathrm{SL}_2(\mathbb{R})$ on $T^1\mathbf{H}$. Prove that this action is transitive. Prove also that the stabilizer of any fixed vector $v \in T^1\mathbf{H}$ equals $\{\pm I\}$. Note that if we make a choice of a fixed 'base vector' $v_0 \in T^1\mathbf{H}$, then it follows that the map $g \mapsto g(v_0)$, $\mathrm{SL}_2(\mathbb{R}) \to T^1\mathbf{H}$ induces an identification of the Lie group $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ with $T^1\mathbf{H}$. The standard choice is to let v_0 be the upward unit vector at the point $i \in \mathbf{H}$.

(c) It is well-known that the geodesics in $\mathbf{H}$ are semicircles (or lines) that are orthogonal to the real axis. Any $v \in T^1\mathbf{H}$ is tangent to a unique geodesic G_v. The *geodesic flow* on $T^1\mathbf{H}$ moves any $v \in T^1\mathbf{H}$ a distance t along the geodesic G_v. Prove that under the identification in part (b) above, with the standard choice of v_0, the geodesic flow corresponds to the flow

$$g \mapsto g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \qquad \text{on } \ \mathrm{PSL}_2(\mathbb{R}).$$

(d) The *horocycles* in $\mathbf{H}$ are the circles that are tangent to the real axis (and the lines that are parallel to the real axis). Each $v \in T^1\mathbf{H}$ is an inward normal vector to a unique horocycle H_v. The *horocycle flow* on $T^1\mathbf{H}$ moves any $v \in T^1\mathbf{H}$ a distance t (counterclockwise, for $t > 0$) along H_v. Prove that under the identification in part (b) above, with the standard choice of

v_0, the horocycle flow corresponds to the flow

$$g \mapsto g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \text{on } \mathrm{PSL}_2(\mathbb{R}).$$

(e) Let X be any compact, connected surface of constant negative curvature -1. It is known that there exists a covering map $\rho : \mathbf{H} \to X$ that is a local isometry. Let

$$G = \mathrm{SL}_2(\mathbb{R}) \quad \text{and} \quad \Gamma = \{\gamma \in \mathrm{SL}_2(\mathbb{R}) \,:\, \rho \circ \gamma \equiv \rho\}.$$

Prove that Γ is a discrete subgroup of G and that $\Gamma\backslash G$ is compact. We have $-I \in \Gamma$; hence there is a natural identification $\Gamma\backslash G = \widetilde{\Gamma}\backslash \mathrm{PSL}_2(\mathbb{R})$ where $\widetilde{\Gamma}$ is the image of Γ in $\mathrm{PSL}_2(\mathbb{R})$. Prove also that the unit tangent bundle T^1X can be identified with $\Gamma\backslash G$ in such a way that the geodesic flow on T^1X corresponds to the flow

$$\Gamma g \mapsto \Gamma g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

on $\Gamma\backslash G$, and the horocycle flow on T^1X corresponds to the flow

$$\Gamma g \mapsto \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on $\Gamma\backslash G$.

Problem 2. *Basics about homogeneous spaces* $\Gamma\backslash G$

Let G be a Lie group[1] let Γ be a discrete subgroup of G, set $X = \Gamma\backslash G$, and let $\pi : G \to X$ be the projection map; $\pi(g) = \Gamma g$. Let μ be a left Haar measure on G. Let $F \subset G$ be a Borel set which is a *fundamental domain* for $\Gamma\backslash G$ (that is, $G = \bigcup_{\gamma\in\Gamma} \gamma F$ and $\gamma F \cap \gamma' F = \emptyset$ for any two $\gamma \neq \gamma'$ in Γ). [2]

(a) Prove that we obtain a Borel measure μ_X on X by setting $\mu_X(E) := \mu(\pi^{-1}(E) \cap F)$ for every Borel subset $E \subset X$.

(b) Prove that μ_X is independent of the choice of F.

(c) Prove that μ can be expressed in terms of μ_X by the formula

$$\int_G f \, d\mu = \int_X \sum_{g \in \pi^{-1}(x)} f(g)\, d\mu_X(x), \qquad \forall f \in L^1(G, \mu).$$

In particular $\mu(E) = \int_X \#(\pi^{-1}(x) \cap E)\, d\mu_X(x)$ for every Borel set $E \subset G$.

(d) Prove that if $\mu_X(X) < \infty$ then μ is right invariant, viz., G is unimodular.

[1] Or more generally, let G be a second countable locally compact group.

[2] It is an interesting exercise to *prove* that such a set F always exists.

Problem 3. *Sobolev norms on $X = \Gamma\backslash G$; (very) basic properties.*
(See, e.g., [10, Sec. 2.9.1].)

Let Γ be a lattice in a Lie group G, and let $X = \Gamma\backslash G$, and let μ be the G-invariant probability measure on X (as usual). Let $\mathfrak{g}$ be the Lie algebra of G, and fix a linear basis $\mathcal{B}$ of $\mathfrak{g}$. For $k \in \mathbb{Z}_{\geq 0}$, $f \in C^k(X)$ and $1 \leq p \leq \infty$, put

$$S_{p,k}(f) = \sum_{\mathrm{ord}(\mathcal{D}) \leq k} \|\mathcal{D}f\|_{L^p},$$

where $\mathcal{D}$ runs through all monomials in $\mathcal{B}$ of order $\leq k$, and $\mathcal{D}$ acts on f by right differentiation (so that, in particular, $Yf(g) = \frac{d}{dt} f(g\exp(tY))\big|_{t=0}$ for any $Y \in \mathcal{B}$); furthermore, $\|F\|_{L^p} := \left(\int_X |F|^p \, d\mu\right)^{1/p}$ for any $F : X \to \mathbb{C}$. Note that we may have $S_{p,k}(f) = +\infty$.

(a). Prove that changing $\mathcal{B}$ only distorts $S_{p,k}$ by a bounded factor. That is, if $S'_{p,k}$ is the norm obtained by replacing $\mathcal{B}$ by another basis, then there exist constants $0 < c_1 < c_2$ (which may depend on k) such that

$$c_1 S_{p,k}(f) \leq S'_{p,k}(f) \leq c_2 S_{p,k}(f) \qquad \text{for all } f \in C^k(X).$$

(b). For $g \in G$ and $f \in C^k(X)$, define $T_g f \in C^k(X)$ through $[T_g f](x) = f(xg)$ $(x \in X)$. Prove that for every compact subset $F \subset G$, there exists a constant $C = C(F, k) > 0$, such that

$$S_{p,k}(T_g f) \leq C \cdot S_{p,k}(f), \qquad \forall f \in C^k(X),\ g \in F.$$

Problem 4. *Polynomial versus exponential divergence.* Let Γ be a lattice in a Lie group G. Recall that for any one-parameter subgroup (h_t) of G we have $h_t^{-1}(\exp X)h_t = \exp\big(\mathrm{Ad}_{h_{-t}}(X)\big)$ for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$, and hence

$$\Gamma g(\exp X)h_t = \Gamma g h_t \exp\big(\mathrm{Ad}_{h_{-t}}(X)\big) \qquad (\forall g \in G,\ X \in \mathfrak{g},\ t \in \mathbb{R}).$$

(a) Prove that if (h_t) is unipotent ($\overset{\text{def}}{\Leftrightarrow}$ Ad_{h_t} is unipotent, $\forall t$), then there is a linear basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ for $\mathfrak{g}$ ($n = \dim G = \dim \mathfrak{g}$) and a *strictly upper triangular* matrix $U \in M_n(\mathbb{R})$ such that

$$\mathrm{Ad}_{h_t} = e^{tU}, \qquad \forall t \in \mathbb{R},$$

in the basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$. Note also that the matrix e^{tU} has the form $(p_{i,j}(t))_{i,j=1,\ldots,n}$ where $p_{i,j}(t) \equiv 0$ for all $1 \leq j < i \leq n$, $p_{i,i}(t) \equiv 1$ for all $1 \leq i \leq n$, and $p_{i,j}(t)$ is a polynomial of degree $\leq j - i$ for all $1 \leq i < j \leq n$.

(b) Prove that if (h_t) is diagonal ($\overset{\text{def}}{\Leftrightarrow}$ Ad_{h_t} is diagonalizable over $\mathbb{R}$, $\forall t$), then there is a linear basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ for $\mathfrak{g}$ ($n = \dim G = \dim \mathfrak{g}$) and constants $c_1, \ldots, c_n \in \mathbb{R}$

$$\mathrm{Ad}_{h_t} = \mathrm{diag}[e^{c_1 t}, \cdots, e^{c_n t}], \qquad \forall t \in \mathbb{R},$$

in the basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$.

Hint for (a) and (b): Having fixed a basis of $\mathfrak{g}$, we have that $t \mapsto \mathrm{Ad}_{h_t}$ is a one-parameter subgroup of $\mathrm{GL}_n(\mathbb{R})$; hence there exists a matrix $A \in M_n(\mathbb{R})$ such that $\mathrm{Ad}_{h_t} = e^{tA}$, $\forall t \in \mathbb{R}$. The task is to prove that the basis of $\mathfrak{g}$ can be chosen so that A is upper triangular (in (a)), resp., diagonal (in (b)). One approach is to consider the Jordan normal form of A (apriori this requires passing to $\mathbb{C}^n$).

(c) Assume that X is *compact.*[3] Use (a) to prove that there exists an upper bound on the possible polynomial rate of mixing of the unipotent flow (h_t) on X. Namely: Given $\ell \in \mathbb{Z}^+$, there exists a constant $d_{\max}$ (which only depends on ℓ and the dimension of G) such that if C and d are any real positive constants with the property that

$$\left| \int_X f_1(xh_t) f_2(x)\, d\mu(x) - \mu(f_1)\mu(f_2) \right| \leq C \cdot S_{2,\ell}(f_1) \cdot S_{2,\ell}(f_2) \cdot t^{-d},$$
$$\forall f_1, f_2 \in C^\infty(X),\ t \geq 1,$$

then $d \leq d_{\max}$.

[3] Can you get rid of this assumption?

Problem 5. *Equidistribution of an expanding horocycles; proof using Margulis' thickening technique.*

Let $G = \mathrm{SL}_2(\mathbb{R})$, let Γ be a lattice in G and set $X = \Gamma\backslash G$. Following the outline in the lecture, prove that for any $p_0 \in X$ and $f \in C_c(X)$,

$$\int_0^1 f(p_0 u_s a_t)\, ds \to \mu(f) \qquad \text{as } t \to -\infty. \tag{1}$$

Hint: Things become technically easier if we don't define f_1 literally as in the lecture, but instead *lift* the horocycle to $G = \mathrm{SL}_2(\mathbb{R})$ and consider an ε-neighbourhood there. Thus: Fix some $g_0 \in G$ so that $p_0 = \Gamma g_0$, and for $\varepsilon > 0$ small, define the function $\widetilde{f}_1 : G \to \{0, 1\}$ by $\widetilde{f}_1(g) = \chi_I(x)\chi_\varepsilon(y)\chi_\varepsilon(z)$ if $g = g_0 u_x\, a_y\, \widetilde{u}_z$ and $\widetilde{f}_1(g) = 0$ if $g \notin g_0 G_+$; this function $\widetilde{f}_1$ is clearly the characteristic function of an "ε-neighbourhood" of the horocycle $\{g_0 u_s \,:\, s \in [0, 1]\}$ in G. Finally define $f_1 : X \to \mathbb{Z}_{\geq 0}$ through $f_1(p) = \sum_{g \in \pi^{-1}(p)} \widetilde{f}_1(g)$ ($p \in X$), where $\pi : G \to X = \Gamma\backslash G$ is the standard projection. Now by "unfolding" we have $\int_X f_1(xa_{-t}) f_2(x)\, d\mu(x) = \int_X f_1(x) f_2(xa_t)\, d\mu(x) = \int_G \widetilde{f}_1(g) f_2(ga_t)\, d\mu(g)$, which can be analyzed further using the explicit definition of $\widetilde{f}_1$.

Problem 6. *"Ratner $\Rightarrow$ Weyl equidistribution"*

Let X be the torus $X = \mathbb{R}^n/\mathbb{Z}^n$ (or "$\mathbb{Z}^n\backslash\mathbb{R}^n$", if you prefer), let $\boldsymbol{v} \in \mathbb{R}^n$, and let Φ_t be the following flow on X:

$$\Phi_t(\boldsymbol{x}) = \boldsymbol{x} + t\boldsymbol{v} \qquad (\boldsymbol{x} \in \mathbb{R}^n/\mathbb{Z}^n,\ t \in \mathbb{R}).$$

Verify that Ratner's Theorem implies, as a very special case, the following fact: If ν is a Φ_t-invariant ergodic Borel probability measure on X, then there exists a point $\boldsymbol{x} \in X$ and a linear subspace $V \subset \mathbb{R}^n$ such that $\boldsymbol{v} \in V$ and $V \cap \mathbb{Z}^n$ contains a basis of V [4], the closure of the orbit $\{\Phi_t(\boldsymbol{x}) \,:\, t \in \mathbb{R}\}$ in X equals $\boldsymbol{x} + X_V$ where $X_V := V/(V \cap \mathbb{Z}^n)$, and ν is the unique V-invariant probability measure on $\boldsymbol{x} + X_V$.

(Of course, you may also like to verify the above statement in a more elementary way, e.g. using Fourier analysis. You may like to focus in particular on the case when $\boldsymbol{v}$ is such that $\boldsymbol{v} \cdot \boldsymbol{m} \neq 0$ [5] for all $\boldsymbol{m} \in \mathbb{Z}^n$; then $V = \mathbb{R}^n$ and $X_V = X$, so that the conclusion is that the flow Φ_t is uniquely ergodic.)

[4] When this holds, one says that V is a *rational* linear subspace of $\mathbb{R}^n$; note that it implies that $V \cap \mathbb{Z}^n$ is a lattice in V (of rank $= \dim V$), and thus $X_V := V/(V \cap \mathbb{Z}^n)$ is a torus, namely a subtorus of $\mathbb{R}^n/\mathbb{Z}^n$ of dimension $\dim V$.

[5] Here "$\cdot$" denotes the standard scalar product in $\mathbb{R}^n$, viz., $\boldsymbol{v} \cdot \boldsymbol{m} = \sum_{j=1}^n v_j m_j$.

Problem 7. *Interpreting the statement of Ratner's Theorem in another (very) special case.*

Prove that in the special case $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $u_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, Ratner's Theorem implies that if $\nu \in P(X)$ is Φ_t-invariant and ergodic then either $\nu = \mu_X$ (the unique G-invariant probability measure on X) or $\nu = \lambda_y$ for some $y > 0$, where λ_y is the uniform probability measure along "the closed horocycle at height y", that is,

$$\lambda_y(f) = \int_0^1 f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}\right) dx \qquad (\forall f \in C_c(X)).$$

(This fact was first proved by Dani, 1978.)