Quantum Ergodicity and L-functions

Problem Session and discussion 2

Topics for discussion

1. What goes wrong if you try to use the approach of Luo–Sarnak for the measure $d\mu_n = |u_n(z)|^2\, d\mu(z)$?

Exercises

1. Explain the proof of van der Corput's theorem below (from Graham–Kolesnik):

   Theorem 1. *Suppose that f is real valued with two continuous derivatives on $I = (a, b]$. Moreover, suppose that for some constant $\lambda > 0$ and some $\alpha \geq 1$ we have*

   $$\lambda \leq |f'(x)| \leq \alpha\lambda, \quad x \in I.$$

   Then

   $$\sum_{n \in I} e(f(n)) \ll \alpha|I|\lambda^{1/2} + \lambda^{-1/2}.$$

   Proof. Let $\delta < 1/2$ be a parameter to be chosen later. From the hypothesis, it follows that I can be split into $\leq \alpha|I|\lambda + 2$ intervals on which $||f'|| \geq \delta$, and $\leq \alpha|I| + 1$ other intervals, each of length $\leq 2\delta/\lambda$. We apply the Kuzmin–Landau Theorem to the former set of intervals, and the trivial estimate to the latter. We get

   $$\sum_{n \in I} e(f(n)) \ll (\alpha|I|\lambda + 1)(1/\delta + \delta/\lambda + 1).$$

   We choose $\delta = \lambda^{1/2}$. This proves the desired bound if $\lambda \leq 1/4$. If $\lambda > 1/4$, the result follows from the trivial estimate. □

2. Let $\mu(\sigma)$ denote the infimum of the set of real numbers θ such that

   $$|\zeta(\sigma + it)| \ll t^{\theta}.$$

   (a) Let $\delta > 0$. Directly prove that for $\sigma \geq 1 + \delta$ and $t \in \mathbb{R}$

   $$\zeta(\sigma + it) \gg 1,$$

   where the implied constant depends at most on δ.

(b) For $\sigma = 1$ the proof of the prime number theorem due to Hadamard and de la Vallée Poussin gives

$$\zeta(1+it) \gg \frac{1}{\log t}$$

for $t \geq 2$.

Directly prove $\zeta(1+it) \ll \log t$, $t \geq 2$. Explain why these results imply $\mu(\sigma) = 0$ for $\sigma \geq 1$.

(c) Use the functional equation

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(\frac{s}{2})} \zeta(1-s)$$

and Stirling's formula to show that $\mu(\sigma) = 1/2 - \sigma$ for $\sigma \leq 0$.

The following result is an analogue of the maximum modulus principle from complex analysis for unbounded regions.

Theorem (Phragmén–Lindelöf) Let f be holomorphic in the region

$$\Omega = \{\sigma + it : \sigma_1 \leq \sigma \leq \sigma_2\}.$$

Suppose that for all $\varepsilon > 0$ we have

$$|f(s)| \ll \exp(\varepsilon |t|), \quad \forall s \in \Omega.$$

Assume that there exist constants $c_1, c_2 > 0$ such that

$$|f(\sigma_1 + it)| \ll (1+|t|)^{c_1}, \qquad |f(\sigma_2 + it)| \ll (1+|t|)^{c_2}.$$

Then

$$|f(\sigma + it)| \ll (1+|t|)^{c(\sigma)}, \quad \forall s = \sigma + it \in \Omega,$$

where $c(\sigma)$ is a linear function of σ with $c(\sigma_1) = c_1$ and $c(\sigma_2) = c_2$.

(d) Use the above theorem to show that $\mu(\sigma)$ is a convex function. Deduce that

$$\mu(1/2) \leq \frac{1}{4},$$

i.e. $|\zeta(1/2 + it)| \ll t^{1/4+\varepsilon}$. This is called the *convexity bound.*