Quantum Ergodicity and L-functions

Problem Session and discussion 1

Topics for discussion

1. How to prove the Fourier expansion of the non-holomorphic Eisenstein series in Notes 1(2.5).
2. Why is $T_n f$ automorphic for Γ?
3. Recall the Poisson summation formula.
4. Why is the Laplace operator commuting with linear fractional transformations?

Exercises
Group 1: Convolution L-functions

1. Recall the definition of the arithmetic function $\sigma_a(n) = \sum_{d|n} d^a$. Let $k \in \mathbb{C}$. Prove that for $\Re(s) > \Re(k) + 1$ we have
$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s-k)\zeta(s).$$
2. Show the Ramanujan identity
$$\sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)\left(\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta}\right) T^n = \frac{1 - \alpha\beta\gamma\delta T^2}{(1-\alpha\gamma T)(1-\alpha\delta T)(1-\beta\gamma T)(1-\beta\delta T)},$$
for $\alpha \neq \beta$, $\gamma \neq \delta$ and
$$|T| < \frac{1}{\max(|\alpha\gamma|, \alpha\delta|, |\beta\gamma|, |\beta\delta|)}.$$
Let $a, b \in \mathbb{C}$. Prove the Ramanujan formula:
$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$
for
$$\mathrm{Re}(s) > \max\{1, \mathrm{Re}(a)+1, \mathrm{Re}(b)+1, \mathrm{Re}(a+b)+1\}.$$

3. Let $u_j(z)$ be an even Hecke eigenform and let $\lambda_j(n)$ be its Hecke eigenvalues. Show that the nth Fourier coefficients of u_j are proportional to $\lambda_j(n)$ i.e. we can write its Fourier decomposition as
$$u_j(z) = \rho_j(1) \sum_{n\geq 1} \lambda_j(n) y^{1/2} K_{it_j}(2\pi n y)\cos(2\pi n x),$$
where $\rho_j(1)$ is a normalisation constant so that $\|u_j\| = 1$.

   Recall that the quadratic polynomial $1 - \lambda_j(p)X + X^2$ can be factored as $(1 - \alpha_j(p)X)(1 - \beta_j(p)X)$ with
$$\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1.$$
   Show that
$$\lambda_j(p^k) = \sum_{0\leq l\leq k} \alpha_j(p)^l \beta_j(p)^{k-l}.$$

4. With the same notation as above, let $R_{j,b}(s)$ be defined by
$$R_{j,b}(s) = \sum_{n=1}^{\infty} \frac{\lambda_j(n) n^{ib} \sigma_{-2ib}(n)}{n^s}.$$
   Directly show (without the general result on Euler factors of Rankin–Selberg convolutions) that
$$R_{j,b}(s) = \frac{1}{\zeta(2s)} L(u_j, s - ib) L(u_j, s + ib).$$

5. Let f and g be two automorphic forms, which are Hecke eigenfunctions with systems of Hecke eigenvalues $\lambda_j(n)$ and $\lambda_m(n)$, and let the Rankin–Selberg L-function be defined by:
$$L(f \otimes g, s) = \zeta(2s) \sum_{n\geq 1} \frac{\lambda_j(n)\lambda_m(n)}{n^s}.$$
   Without using the proof in Lemma 1.6.1 in D. Bump: Automorphic forms and representations, show that
$$\begin{aligned} L(f\otimes g, s) = \prod_p &(1 - \alpha_j(p)\alpha_m(p)p^{-s})^{-1}(1 - \alpha_j(p)\beta_m(p)p^{-s})^{-1} \\ &\times (1 - \beta_j(p)\alpha_m(p)p^{-s})^{-1}(1 - \beta_j(p)\beta_m(p)p^{-s})^{-1}. \end{aligned}$$

Group 2: Euler summation and Stirling's formula

1. Let ψ be function defined as:
$$\psi(x) = x - [x] - \tfrac{1}{2},$$
   where $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ is the floor function.

(a) Prove that for $x \notin \mathbb{Z}$

$$\psi(x) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}.$$

Hint: You may use the fact from Fourier analysis that if f is of bounded variation on the unit interval then

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\widehat{f}(n)e(nx) = \frac{f(x^+)+f(x^-)}{2}$$

with $f(x^\pm)$ the one-sided limits of f at x.

(b) Use part (a) to show that

$$\zeta(2) := \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

(c) Let $f \in C^1([a,b])$. Prove

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,dx + \int_a^b \psi(x)f'(x)\,dx + \psi(a)f(a) - \psi(b)f(b).$$

Hint: Express the sum as a Stieltjes integral and integrate by parts.

(d) Use part (c) to show that

$$\log n! = n\log n - n + O(\log n).$$

2. While the previous exercise provides the behaviour of $n! = \Gamma(n+1)$, we need the behaviour of the Gamma function on vertical lines. Recall Stirling's formula

$$\log\Gamma(s) = s\log s - s + \frac{1}{2}\log(2\pi) + O_\delta(|s|^{-1})$$

for $-\pi+\delta < \arg(s) < \pi-\delta$.

Prove that

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|).$$

uniformly for $-\pi+\delta \le \arg s \le \pi-\delta$. Deduce that

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-it\right) = \log\left(\frac{1}{4}+t^2\right) + O(t^{-1})$$

for $t \ge 1$.